IEOR E4004: Deterministic Models

Jay Sethuraman

Midterm (sample)

1. (15 points) Consider the linear programming problem

$$\begin{aligned} \text{Max} \quad & x_1 + 3x_2 - x_3 \\ \text{subject to:} \quad & \\ & 2x_1 + 2x_2 - x_3 \leq 10 \\ & 3x_1 - 2x_2 + x_3 \leq 10 \\ & x_1 - 3x_2 + x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   (a) (5 points) Write down the LP in standard form (equality constraints, non-negative variables).

   (b) (10 points) Solve the LP using the simplex method.

2. (15 points) Suppose you are given a set of n observations

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}.$$

   You are asked to fit a straight line of the form

$$y \ = \ bx + c$$

   to this data. You are asked to find values of b and c so as to get the best possible fit, described below.

   For any values of b and c, the error of the ith observation is the absolute value of the difference between y_i and $bx_i + c$. Suppose the objective is to minimize the maximum error among the n observations.

   (a) (5 points) Formulate this problem as a *linear* program, clearly indicating what your decision variables are.

   (b) (10 points) Suppose instead you are asked to fit a function of the form

$$y \ = \ ax^2 + bx + c$$

   to this data, and the objective is to minimize the sum of all errors. Can this be formulated as a linear program? If so, provide such a formulation; if not, explain why not.

3. (20 points; 2 points for each part) Consider the linear programming problem

$$\text{Max} \quad 8x_1 + 6x_2$$

subject to:

$$\begin{aligned} &3x_1 + 2x_2 \le 28, \\ &5x_1 + 2x_2 \le 42, \\ &x_1 \le 8, \\ &x_2 \le 8, \\ &x_1, x_2 \ge 0. \end{aligned}$$

(i) Does this LP have a finite optimal value? Justify briefly.

(ii) Write down the dual of this LP. Does the dual problem have a finite optimal value?

(iii) Use the complementary slackness (CS) conditions to show that the solution $x_1 = 8$, $x_2 = 1$ cannot be optimal.

(iv) Consider the solution $x_1 = 4$, $x_2 = 8$, with value 80. Show that this is optimal by constructing a dual feasible solution with objective value 80.

(v) Find the optimal primal dictionary without doing any simplex iterations. (Hint: x_1 and x_2 are basic variables in an optimal solution.)

(vi) Suppose the constraint $x_1 + x_2 \le 11$ is added to the primal LP. Does the optimal solution change? (You don't need to find the new solution.) If so, does the optimal value increase or decrease?

(vii) Suppose the constraint $x_2 \le 8$ is changed to $x_2 \le 8 + \delta$. Find the range of δ for which the optimal basis does not change.

(viii) Suppose the constraint $5x_1 + 2x_2 \le 42$ is changed to $5x_1 + 2x_2 \le 42 + \delta$. Find the range of δ for which the optimal basis does not change.

(ix) Find the range of c_1 (which is currently 8) for which the optimal basis does not change.

(x) Suppose a third party were to offer you the option of increasing the right hand side of the first constraint from 28 to 30 for a lump-sum payment of $\$K$. For what values of K would you find this offer attractive?

Period	Time	Number needed	Pay rate (per hour)
1	12 midnight -4 am	20	13
2	4-8 am	30	12
3	8-12 noon	100	8
4	12-4 pm	120	8
5	4-8 pm	90	9
6	8-12 midnight	50	11

Figure 1: Staffing Requirements

4. (20 points; 5 for each part) Consider the following dictionary:

$$
\begin{array}{rcl}
s_1 & = & 4 - x_1 - x_2 - x_4 \\
s_2 & = & A + x_2 + x_3 - x_4 \\
s_3 & = & 2 - x_1 + x_3 - 2x_4 \\
\hline
z & = & Bx_1 - x_2 - Cx_3 - 3x_4
\end{array}
$$

Find ranges for the values of A, B, and C such that the following statements are valid:

(a) The solution $x_1 = x_2 = x_3 = x_4 = 0$ is the unique optimal solution.

(b) The unique optimal solution has $x_1 > 0$ and $x_2 = x_3 = x_4 = 0$.

(c) There is an optimal solution with $x_1 > 0$ and $x_2 = x_3 = x_4 = 0$, but it is not unique.

(d) The problem has an unbounded optimal objective value.

5. (10 points) The staffing requirements and the rate of pay for each of six shifts during a 24-hour day is shown in Figure 1.

Each employee works during a 3-period shift with the middle period off. For example an employee that starts work at 4am works periods 2 and 4. Note that period 1 follows immediately after period 6, i.e., an employee that start work at 8pm (period 6) has a break during period 1 and continues working during period 2. Formulate the problem of finding a daily schedule of employees that minimizes cost while providing a sufficient number of employees for each period as a linear program, ignoring integrality requirements. Clearly define all variables.