Sample exam (Solutions)

1.

(a) $\text{Max } x_1 + 3x_2 - x_3$

<u>Subject to:</u>

$$2x_1 + 2x_2 - x_3 + s_1 = 10$$
$$3x_1 - 2x_2 + x_3 + s_2 = 10$$
$$x_1 - 3x_2 + x_3 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

(b) Initial dictionary:

$$s_1 = 10 - 2x_1 - 2x_2 + x_3$$
$$s_2 = 10 - 3x_1 + 2x_2 - x_3$$
$$s_3 = 10 - x_1 + 3x_2 - x_3$$
$$\overline{}$$
$$z = x_1 + 3x_2 - x_3$$
$$\uparrow$$

x_2 enters; s_1 leaves.

$\Rightarrow$

$$x_2 = 5 - x_1 - \tfrac{1}{2}s_1 + \tfrac{1}{2}x_3$$
$$s_2 = 10 - 3x_1 + 2(5 - x_1 - \tfrac{1}{2}s_1 + \tfrac{1}{2}x_3) - x_3$$
$$= 20 - 5x_1 - s_1$$
$$s_3 = 10 - x_1 + 3(5 - x_1 - \tfrac{1}{2}s_1 + \tfrac{1}{2}x_3) - x_3$$
$$= 25 - 4x_1 - \tfrac{3}{2}s_1 + \tfrac{x_3}{2}$$
$$\overline{}$$
$$z = x_1 + 3(5 - x_1 - \tfrac{1}{2}s_1 + \tfrac{1}{2}x_3) - x_3$$

Simplifying, we get:

$$x_2 = 5 - x_1 - \tfrac{1}{2}s_1 + \tfrac{1}{2}x_3$$

$$s_2 = 20 - 5x_1 - s_1$$

$$s_3 = 25 - 4x_1 - \tfrac{3}{2}s_1 + \tfrac{1}{2}x_3$$

$$z = 15 - 2x_1 - \tfrac{3}{2}s_1 + \tfrac{1}{2}x_3$$

x_3 enters, and can be increased by any amount.

$\Rightarrow$ Problem is unbounded.

Take $\left\{ \begin{array}{c} x_3 = t,\ x_2 = 5 + t/2 \\ x_1 = 0. \end{array} \right\}$ for any $t > 0$

This is feasible & has obj value $15 + t/2$.

2

— (*)

Objective function

(*) as :

— (**)

is now nonlinear.

by

all i. — (***)

than $\max_i |y_i - bx_i - c|$,

be the case because

is still not linear

also be fixed by

all i

all i

$$\min_{\{b,c\}} \left[\max_i \{ |y_i - bx_i - c| \} \right]$$

This is not an LP formulation because the

is nonlinear. The trick is to rewrite

$$\min \ v$$

$$v = \max_i \{ |y_i - bx_i - c| \}$$

This is still not linear as the constraint

<u>But:</u> we could replace that (**) b

$$v \geq |y_i - bx_i - c| \quad \text{for}$$

While this <u>allows</u> v to be <u>strictly bigger</u> in an <u>optimal solution</u> this will never <u>we are trying to minimize v</u>. But this because of the $|\cdot|$ sign. But this can using:

$$v \geq (y_i - bx_i - c), \quad \text{for}$$

$$v \geq -(y_i - bx_i - c), \quad \text{for}$$

So the Solution to (a) is:

$$\text{Min } v$$
$$v \geq y_i - bx_i - c, \quad \text{for all } i$$
$$v \geq bx_i + c - y_i, \quad \text{for all } i$$

(b) Yes, this can be rewritten as a linear programming problem: Here is one formulation

$$\text{Min } \sum_{i=1}^{n} e_i$$

$$e_i \geq (y_i - ax_i^2 - bx_i - c), \quad \text{for all } i$$

$$e_i \geq (ax_i^2 + bx_i + c - y_i), \quad \text{for all } i$$

[decision variables are a, b, c]

3.

(i) Yes. x_1 & x_2 must both be at most 8, so the optimal value $\leq 8 \times 8 + 6 \times 8 = 64 + 48 = 112$.
Also, $x_1 = 0$, $x_2 = 0$ is feasible to the LP.
So the LP is feasible & cannot be unbounded, so it must have an optimal value.

(ii) Min $28y_1 + 42y_2 + 8y_3 + 8y_4$

$$3y_1 + 5y_2 + y_3 \geq 8$$
$$2y_1 + 2y_2 + y_4 \geq 6$$
$$y_1, y_2, y_3, y_4 \geq 0$$

The dual has a finite optimal value by the strong duality theorem.

(iii) $x_1 = 8$, $x_2 = 1$.
Because $3x_1 + 2x_2 = 26 < 28 \Rightarrow y_1 = 0$ (to satisfy CS conditions)

$x_2 < 8 \Rightarrow y_4 = 0$ (to satisfy CS conditions).

If $y_1 = y_4 = 0$, then $y_2 \geq 3$ (to satisfy the second dual constraint).

But this implies $3y_1 + 5y_2 + y_3 \geq 15 > 8 \Rightarrow x_1$ must be zero to satisfy CS conditions.

So $x_1 = 8$, $x_2 = 1$ cannot be optimal

(iv) $x_1 = 4, \quad x_2 = 8. \qquad z = 80$

Let us try to use CS conditions to get information about the dual variables in an optimal dual solution.

(Third constraint) ~~x_1~~ $x_1 = 4 < 8 \Rightarrow y_3 = 0$ (by CS conditions)

(Second constraint) $5x_1 + 2x_2 = 36 < 42 \Rightarrow y_2 = 0$. (by CS condition).

Also: $x_1 > 0 \Rightarrow 3y_1 + 5y_2 + y_3 = 8. \Rightarrow y_1 = 8/3$

$x_2 > 0 \Rightarrow 2y_1 + 2y_2 + y_4 = 6. \Rightarrow y_4 = 2/3$.

So: $[y_1 = 8/3, \; y_4 = 2/3]$ is ~~[illegible]~~ the dual optimal solution. (as it is dual feasible & satisfies CS conditions).

Another Check: ~~do~~ $y_1 = 8/3, \; y_4 = 2/3$, $y_2 = 0 = y_3$ has value

$$28\left(\frac{8}{3}\right) + 8\left(\frac{2}{3}\right) = \frac{240}{3} = 80,$$

Same as the primal objective value.

So the given x solution is primal optimal & the y we found is dual optimal.

(v) We use x_1, x_2, S_2, S_3 as the basic variables.
Use the last equation to get: (S_1, S_4 are non-basic)

$$\boxed{x_2 = 8 - S_4.}$$

From the first equation:

$$3x_1 + 2x_2 + S_1 = 28.$$

$$\Rightarrow 3x_1 = 28 - S_1 - 2x_2 = 28 - S_1 - 2(8 - S_4)$$

$$= 12 - S_1 + 2S_4$$

$$\Rightarrow \boxed{x_1 = 4 - \frac{1}{3}S_1 + \frac{2}{3}S_4}$$

$$S_2 = 42 - 5\left(4 - \frac{1}{3}S_1 + \frac{2}{3}S_4\right) - 2(8 - S_4)$$

$$= 42 - 20 + \frac{5}{3}S_1 - \frac{10}{3}S_4 - 16 + 2S_4, \quad \text{so}$$

$$\boxed{S_2 = 6 + \frac{5}{3}S_1 - \frac{4}{3}S_4}$$

$$S_3 = 8 - x_1 = 4 + \frac{1}{3}S_1 - \frac{2}{3}S_4$$

$$z = 8x_1 + 6x_2 = 32 - \frac{8}{3}S_1 + \frac{16}{3}S_4 + 48 - 6S_4$$

$$= 80 - \frac{8}{3}S_1 - \frac{2}{3}S_4.$$

So the optimal primal dictionary is:

$$
\begin{aligned}
x_2 &= 8 - s_4 \\
x_1 &= 4 - \tfrac{1}{3}s_1 + \tfrac{2}{3}s_4 \\
s_2 &= 6 + \tfrac{5}{3}s_1 - \tfrac{4}{3}s_4 \\
s_3 &= 4 + \tfrac{1}{3}s_1 - \tfrac{2}{3}s_4 \\
\hline
z &= 80 - \tfrac{8}{3}s_1 - \tfrac{2}{3}s_4 .
\end{aligned}
$$

(vi) Currently $x_1 + x_2 = 12$, so the current solution will violate the new constraint. ~~The optimal value~~. Therefore the optimal solution will change. The opt. value will decrease [Notice, $x_2 = 8$, $x_1 = 4$ is the unique opt. soln. to the original Problem]

(vii) Range analysis for the fourth constraint. To do this, we look at s_4 coefficients in the dictionary (basic variable equations).

Allowable increase in RHS of constraint 4 $= \min\left\{ 4/(2/3) \right\}$
$= \underline{6}$

Allowable decrease in RHS of constraint 4 $= \min\left\{ \frac{8}{1}, \frac{6}{4/3}, \frac{4}{2/3} \right\}$

$= \min\left\{ 8, 4.5, 6 \right\} = \underline{\underline{4.5}}$

So: $\delta \in [-4.5, 6]$

(viii) Currently $5x_1 + 2x_2 = 36 < 42$.

So. δ can be any number ≥ -6 & the opt. basis will not change. So the answer is $\delta \in [-6, \infty)$

(ix) Suppose c_1 goes from 8 to $8+\delta$. Then the last row of the dictionary becomes:

$$Z = 80 - \frac{8}{3} s_1 - \frac{2}{3} s_4 + \delta x_1$$

$$= 80 - \frac{8}{3} s_1 - \frac{2}{3} s_4 + \delta\left(4 - \frac{1}{3} s_1 + \frac{2}{3} s_4\right)$$

$$= [80 + 4\delta] - \frac{(8+\delta)}{3} s_1 - \left(\frac{2-2\delta}{3}\right) s_4$$

For the current basis to be optimal, we want:

$$\frac{8+\delta}{3} \geq 0 \quad \& \quad \frac{2-2\delta}{3} \geq 0$$

$\Rightarrow \delta \geq -8$, & $\delta \leq 1$.. If $c_1 \in [0, 9]$, the current basis is optimal.

(x) Shadow price of the first constraint $= 8/3$.
The allowable increase can be calculated by looking at the primal dictionary — specifically the coefficient of s_1 in the basic equations of the dictionary. By doing the range analysis, we see that the allowable increase $= \min\left\{\frac{6}{5/3}, \frac{4}{1/3}\right\}$ $= \min\left\{\frac{18}{5}, 12\right\} = 3.6$. As the given change (of 2) is within the range, change in profit $= 2 \times 8/3 = 16/3$.
So accept for any $K \leq 16/3$; reject otherwise.

4

(a) $A \geq 0$ for the current solution to be feasible.

$B < 0$, $C > 0$ for the current soln. to be the unique optimal solution. (If $B = 0$ or $C = 0$, it is easy to check that there are other solutions.)

(b) For $x_1 > 0$ in an unique optimal soln., we must have $B > 0$. Then x_1 enters the basis & S_3 leaves.

Now: $x_1 = 2 - S_3 + x_3 - 2x_4$.

$$
\begin{aligned}
\& \; Z &= B(x_1) - x_2 - Cx_3 - 3x_4 \\
&= B(2 - S_3 + x_3 - 2x_4) - x_2 - Cx_3 - 3x_4 \\
&= 2B - BS_3 + Bx_3 - 2Bx_4 - x_2 - Cx_3 - 3x_4 \\
&= 2B - (C - B)x_3 - BS_3 - (2B + 3)x_4 - x_2
\end{aligned}
$$

For this to be optimal, we need $C \geq B$ & for the opt. soln to be unique $C > B$.

Also, we need $A \geq 0$ for feasibility.

(c) Same as (b) but now $B = C$. Then x_3 has a coefficient of 0 in the Z-row of the dictionary & it is easy to verify that x_3 can be introduced into the basis at a positive level. So the answer here is $B > 0$, $C = B$, $A \geq 0$.

(d) $A \geq 0$, $C < 0$ (x_3 is the only variable that can be unbounded.)

5

X_i = # of employees who begin their shift in period i.
($i = 1, 2, 3, 4, 5, 6$)

Any employee who starts his shift in period i works in period i & period $i+2$, with a break in period $i+1$. (All numbers are interpreted "mod 6": i.e. $7 \equiv 1$, $8 \equiv 2$, etc.)

$$\text{Min} \left\{ \begin{array}{l} 13(x_1 + x_5) + 12(x_2 + x_6) + 8(x_3 + x_1) + 8(x_4 + x_2) \\ \quad + 9(x_5 + x_3) + 11(x_6 + x_4) \end{array} \right\}$$

Subject to:

$$x_1 + x_5 \geq 20$$
$$x_2 + x_6 \geq 30$$
$$x_3 + x_1 \geq 100$$
$$x_4 + x_2 \geq 120$$
$$x_5 + x_3 \geq 90$$
$$x_6 + x_4 \geq 50$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

Note: The objective function should be multiplied by 4, strictly speaking, as the wages given are per hour. But this does not make a difference in terms of the optimal solution.