IEOR E4004: Introduction to Operations Research: Deterministic Models

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Final exam

3 hours; open book/notes; no calculators

1. (20 points) Consider the linear integer programming problem

$$\begin{aligned} \text{Max} \quad & 2x_1 - 4x_2 \\ \text{subject to:} \quad & \\ & 2x_1 + x_2 \le 5, \\ & -4x_1 + 4x_2 \le 5, \\ & x_1, x_2 \ge 0, \text{integer.} \end{aligned}$$

   (a) (6 points) Let s_1 and s_2 be the slack variables associated with the first two constraints respectively. Suppose x_1 and s_2 are basic variables in the optimal solution to the associated linear program. Write down the optimal dictionary and the optimal basis.

   (b) (7 points) List all the "cuts" you can find from the optimal dictionary. (Each such cut is valid for the integer program but cuts away the current optimal LP solution). Find the optimal solution of the LP with all these additional inequalities added. Is the new LP optimal solution optimal for the IP also?

   (c) (7 points) Solve the IP using the branch-and-bound method. Clearly indicate the branch-and-bound tree and your computations. (You may solve the intermediate LPs using any method you like.)

2. (20 points) Consider the uncapacited (directed) network flow problem described as follows:

$$\begin{aligned} V &= \{1, 2, 3, 4, 5\}, \\ E &= \{(1,3), (2,1), (2,4), (4,1), (4,3), (4,5), (5,2)\}, \end{aligned}$$

   and

$$c_{13} = 3, c_{21} = 1, c_{24} = 2, c_{41} = 0, c_{43} = 1, c_{45} = 1, c_{52} = 1.$$

   Let $B = \{(1,3), (2,4), (4,3), (4,5)$ be the set of basic arcs, and let the other arcs be nonbasic. Nodes 1 and 2 have a supply of 2 each; node 3 has a demand of 3 and node 4 has a demand of 1.

   (a) (2 points) Find the flow on all arcs corresponding to the given basic solution. Is this solution a basic feasible solution? Justify briefly.

   (b) (4 points) Find an optimal solution to the dual problem.

   (c) (4 points) Find the reduced cost of each nonbasic arc. Is the current solution optimal?

   (d) (5 points) By how much can we decrease c_{52} and still maintain optimality? What is the optimal cost if c_{52} is decreased further?

   (e) (5 points) What is the optimal cost (as a function of δ) if we change the supplies to $2+\delta$ each, and the demands to $1+2\delta$ and 3 respectively. What is the largest value of δ for which the same basis remains optimal? And what happens if we try to increase δ even further?

3. (20 points) A vendor can set up his truck in one of two locations $\{A, B\}$ each day. His profits on the ith day for location A and B are A_i and B_i respectively. However each time he changes location he incurs a cost of c. Suppose the vendor knows A_i and B_i for $i = 1, 2, \ldots, N$. The vendor wishes to maximize his profit over N days. How will you solve the vendor's problem using dynamic programming? Can this problem be formulated as a shortest path problem in a suitably defined graph?

   (5 point bonus) Suppose he worked at location A on day $i-1$. Let $D = B_i - A_i$. It seems like if D is sufficiently large, it will be optimal to switch to B; and if D is sufficiently small, it will be optimal to continue at A. Can you formalize this? That is, find conditions (in terms of D) under which it is optimal for the vendor to stay at A on day i, and conditions under which it is optimal for her to switch to location B on day i.

4. (20 points)

   (a) Let $G = (V, E)$ be an undirected graph with n nodes. Each node must be colored red, blue, or yellow. We are interested in deciding whether the nodes can be colored so that any two nodes connected by an edge have different colors. Formulate a linear integer programming problem that is feasible if and only if such a coloring is possible.

   (b) Let $G = (V, E)$ be an undirected graph with n nodes. A *clique* is a subset of nodes that is "fully connected," i.e., there is an edge between any pair of nodes in a clique. Formulate the problem of finding a maximum size clique (i.e., one with the largest number of nodes) in a graph as a linear integer programming problem. Your formulation should only use $O(n^2)$ variables and constraints.

5. (20 points) Suppose we have a set of n people and a set of n projects. Let A_i be the set of projects that person i is willing to work on. We wish to assign the people to the projects such that each each person is assigned exactly one project and such that each project is assigned to exactly one person. Such an assignment is called a *perfect matching*. Of course, whether or not a perfect matching is possible depends on the set of projects each person is willing to work on.

   For any subset Y of projects, let P_Y be the set of persons who can work on at least one project in Y. We say that Y is *deficient* if the number of elements in Y is more than the number of elements in P_Y. (Suppose no one other than person 1 is willing to work on projects 1 and 2. Then, if $Y = \{1, 2\}$, $P_Y = \{1\}$, and Y is deficient.)

   Prove that a perfect matching is possible if and only if no subset of projects is *deficient*. Note that you have to prove two statements. (Hint: Use the max-flow min-cut theorem on a suitably defined graph.)