IEOR E4004: Introduction to Operations Research: Deterministic Models

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Sample Final Solutions

Problem 1. Please refer to HW 9.

Problem 2.

(a) $x_{13} = 2, x_{24} = 2, x_{45} = 0, x_{43} = 1$ is a basic feasible solution. This solution has cost equal to 11.

(b) $\pi_1 = 0, \pi_2 = 0, \pi_3 = -3, \pi_4 = -2, \pi_5 = -3$. The cost is 11.

(c) Clearly, by strong duality we have an optimal solution. But let's compute the reduced costs anyway. $\bar{c}_{41} = 2, \bar{c}_{52} = 4, \bar{c}_{21} = 1$. They are all nonnegative, which comes as no surprise.

(d) We can decrease it by as much as $\bar{c}_{52} = 4$. If it goes down any further then the optimal cost will be $-\infty$. This is because the reduced cost of $(5, 2)$ is now negative and we can push as much flow as we want along the cycle $5 - 2 - 4 - 5$.

(e) As a function of δ the optimal cost is: $\sum_i \pi_i b_i = 11 + 4\delta$. The same basis is feasible and therefore optimal as long as $2 + \delta \geq 1 + 2\delta$, i.e. $\delta \leq 1$. This is because the demand of $1 + 2\delta$ at node 4 can only be met by the supply of $2 + \delta$ at node 2. If we increase δ further, our basic solution is infeasible.

Problem 3. Please refer to HW 10.

Problem 4. Please refer to HW 9.

Problem 5. Let's define the following graph $G = (N, E)$. Here $N = \{s, 1, 2, ..., n, p_1, p_2, ..., p_n, t\}$. So we have a source node s, sink node t and a node i for each person i and a node p_j for each project j. Now, we include an edge from s to each node i and an edge from each project p_j to node t. Furthermore let's connect person i to project p_j if and only if $p_j \in A_i$, that is if and only if person i is willing to do project j. Edges (s, i) and (p_j, t) for all $i = 1, 2, .., n$ and $j = 1, 2, ..., n$, are given unit capacities, while all other edges are given infinite capacity.

A feasible integer 0-1 $s-t$ flow in this graph corresponds to a feasible allocation of projects to people. Thus, the value of the maximum $s-t$ flow in this graph will tell us if there is a perfect matching or not. That is, there exists a perfect matching in the graph if and only the value of the maximum flow is n.

Given this context let's proceed to prove the if and only if statement. Let Z be the set of all projects and P be the set of all people. Note $|Z| = |P| = n$.

(a) Assume there exists a deficient subset Y_1 of projects such that $|P_{Y_1}| < |Y_1|$.

In that case pick the deficient subset Y_1. Consider the s-t cut $[S, \bar{S}]$ where $S = s \cup (P - P_{Y_1}) \cup (Z - Y_1)$. This cut has capacity: $|P_{Y_1}| + n - |Y_1| < n$, since $|P_{Y_1}| < |Y_1|$. Hence the min-cut of this graph will have value strictly less than n. By the max-flow min-cut theorem, the value of the max-flow will also be less than n and thus we do not have a perfect matching.

(b) Assume there is no deficient subset.

Then let us consider $s-t$ cuts $[S, \bar{S}]$ in the graph **which do not have infinite capacity**. The only such cuts are of the following type $S = s \cup (P - P_Y) \cup (Z - Y)$. Other cuts, except for the generic $S = s$ or $S = N - t$ which both have capacity n, would definitely have infinite capacity. This is because we would have an infinite capacity edge from at least one of the people in to one of the projects.

Thus, by assumption, the capacity of all non-infinite capacity cuts, defined by the subsets Y, is $|P_Y| + n - |Y| = |P_Y| + n - |Y| \geq n$. Thus, any cut in our graph will have capacity at least n (if not infinite). Thus, the min-cut will have capacity exactly n and the max-flow will be equal to n. This implies the existence of a perfect matching.

This concludes the proof.