4.23 from BHM a)

Dual problem is

$$\begin{array}{ll} \min & \sum_{i=1}^{m} b_i y_i + \sum_{j=1}^{n} u_j w_j \\ \text{s.t.} & \sum_i a_{ij} y_i + w_j \geq c_j \quad \forall\, j \\ & y_i \text{ unrestricted } \forall\, i, \;\; w_j \geq 0 \;\forall\, j \end{array} \tag{18}$$

b) The optimality conditions are

$$\left.\begin{array}{r} \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \forall\, i \\ x_j \leq u_j \quad \forall\, j \\ x_j \geq 0 \quad \forall\, j \end{array}\right\} \quad \text{primal feasible}$$
$$\left.\begin{array}{r} \sum_{i=1}^{m} a_{ij} y_i + w_j \geq c_j \quad \forall\, j \\ w_j \geq 0 \quad \forall\, j, \\ y_i \text{ unrestricted } \forall\, i, \end{array}\right\} \quad \text{dual feasible} \tag{19}$$
$$\left.\begin{array}{r} (u_j - x_j) w_j = 0 \quad \forall\, j \\ (c_j - (w_j + \sum_i a_{ij} y_i)) x_j = 0 \quad \forall\, j \end{array}\right\} \quad \text{complementary slackness}$$

c) Define $\bar{c}_j = c_j - \sum_i a_{ij} y_i$. Then the optimality conditions in part b imply

$$\begin{aligned} w_j \geq \bar{c}_j \quad \forall\ j \quad & (1) \\ w_j \geq 0 \quad \forall\ i, \quad & (2) \\ (u_j - x_j) w_j = 0 \quad \forall\ j \quad & (3) \\ (\bar{c}_j - w_j) x_j = 0 \quad \forall\ j \quad & (4) \end{aligned} \tag{20}$$

If $x_j = 0 \Rightarrow x_j < u_j \Rightarrow$ (3) implies $w_j = 0$ and (1) implies $\bar{c}_j \leq 0$.
If $x_j = u_j \Rightarrow x_j > 0 \Rightarrow$ (4) implies $w_j = \bar{c}_j$ and (2) implies $\bar{c}_j \geq 0$.
If $0 < x_j < u_j \Rightarrow$ (3) and (4) implies $w_j = 0$ and $\bar{c}_j = w_j$ and so $\bar{c}_j = 0$.

4.24 from BHM a)

Dual of the original LP is

$$\begin{array}{rc} \min & \sum_{i=1}^m b_i y_i + \sum_{i=1}^q w_i d_i \\ \text{s.t.} & \sum_{i=1}^m y_i a_{ij} + \sum_{i=1}^q w_i e_{ij} \geq c_j \quad \forall j \\ & y_i \geq 0, w_i \geq 0 \end{array} \tag{21}$$

The optimality conditions for the original LP are

$$\text{(Cond 1)} \qquad \begin{array}{c} \left.\begin{array}{c} \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1, ..., m \\ \sum_{j=1}^n e_{ij} x_j \leq d_i \quad i = 1, ..., q \\ x_j \geq 0 \quad \forall\ j \end{array}\right\} \quad \text{primal feasible} \\ \left.\begin{array}{c} \sum_{i=1}^m y_i a_{ij} + \sum_{i=1}^q w_i e_{ij} \geq c_j \quad \forall j \\ y_i \geq 0, w_i \geq 0 \end{array}\right\} \quad \text{dual feasible} \\ \left.\begin{array}{c} (-\sum_{j=1}^n a_{ij} x_j + b_i) y_i = 0 \quad i = 1, ..., m \\ (-\sum_{j=1}^n e_{ij} x_j + d_i) w_i = 0 \quad i = 1, ..., q \\ (-\sum_{i=1}^m y_i a_{ij} - \sum_{i=1}^q w_i e_{ij} + c_j) x_j = 0 \quad \forall j \end{array}\right\} \quad \text{complementary slackness} \end{array} \tag{22}$$

Next consider the Lagrangean problem. Define $\bar{c}_j = c_j - \sum_i a_{ij} y_i$. The dual of the this problem is

$$\begin{array}{rc} \min & \sum_{i=1}^q w_i d_i \\ \text{s.t.} & \sum_{i=1}^q w_i e_{ij} \geq \bar{c}_j \quad \forall j \\ & w_i \geq 0 \end{array} \tag{23}$$

The optimality conditions for the Lagrangean problem are

$$
\text{(Cond 2)} \quad \left.\begin{array}{c} \sum_{j=1}^{n} e_{ij}x_j \le d_i \quad i=1,...,q \\ x_j \ge 0 \;\; \forall\, j \end{array}\right\} \text{primal feasible} \quad \left.\begin{array}{c} \sum_{i=1}^{q} w_i e_{ij} \ge \bar{c}_j \;\; \forall j \\ w_i \ge 0 \end{array}\right\} \text{dual feasible} \quad \left.\begin{array}{c} (-\sum_{j=1}^{n} e_{ij}x_j + d_i)w_i = 0 \quad i=1,...,q \\ (-\sum_{i=1}^{q} w_i e_{ij} + \bar{c}_j)x_j = 0 \;\; \forall j \end{array}\right\} \text{complementary slackness} \tag{24}
$$

It is easy to see that if x_j^*, y_i, w_i satisfy Cond 1, then they also satisfy Cond 2, which implies that x_j^* also solves the Lagrangean problem.

4.30 from BHM Firstly, let's solve

$$
\min_{y_j \ge 0} f(x_1, \ldots, x_n; y_1, \ldots, y_m) \tag{25}
$$

and

$$
\max_{x_j \ge 0} f(x_1, \ldots, x_n; y_1, \ldots, y_m). \tag{26}
$$

We get

$$
\min_{y_j \ge 0} f(x_1, \ldots, x_n; y_1, \ldots, y_m) = \begin{cases} \sum_{j=1}^{n} c_j x_j, & \text{if } -\sum_{j=1}^{n} a_{ij}x_j + b_i \ge 0, \forall i, \\ -\infty, & \text{otherwise,} \end{cases} \tag{27}
$$

and

$$
\max_{x_j \ge 0} f(x_1, \ldots, x_n; y_1, \ldots, y_m) = \begin{cases} \sum_{i=1}^{m} b_i y_i, & \text{if } c_j - \sum_{i=1}^{m} a_{ij}y_i \le 0, \forall j \\ +\infty, & \text{otherwise.} \end{cases} \tag{28}
$$

So

$$
\max_{x_j \ge 0} \min_{y_j \ge 0} f(x_1, \ldots, x_n; y_1, \ldots, y_m) = \min_{y_j \ge 0} \max_{x_j \ge 0} f(x_1, \ldots, x_n; y_1, \ldots, y_m) \tag{29}
$$

implies that the following two problems have the same optimal objective function value:

$$
\begin{array}{ll} \max & \sum_{j=1}^{n} c_j x_j \\ s.t. & -\sum_{j=1}^{n} a_{ij}x_j + b_i \ge 0, \forall i \\ & x_j \ge 0, \forall j, \end{array} \tag{30}
$$

$$\begin{array}{ll} \min & \sum_{i=1}^{m} b_i y_i \\ s.t. & c_j - \sum_{i=1}^{m} a_{ij} y_i \leq 0, \forall j \\ & y_i \geq 0, \forall j. \end{array} \tag{31}$$

However, note that (31) is just the dual problem to (30), so they have same optimal function value just implies the strong duality property.