Recitation #4

2.18 Frequently linear programs are formulated with intervals constraints of the form:

$$5 \leq 6 \cdot x_1 - x_2 + 3 \cdot x_3 \leq 8$$

a. Show that this constraint is equivalent to the constraints

$$\begin{array}{l} 6 \cdot x_1 - x_2 + 3 \cdot x_3 + x_4 = 8 \\ 0 \leq x_4 \leq 3 \end{array}$$

The given interval constraint can be written in two constraints as:

$$5 \leq 6 \cdot x_1 - x_2 + 3 \cdot x_3$$

$$6 \cdot x_1 - x_2 + 3 \cdot x_3 \leq 8$$

And adding slack and surplus variables respectively we have:

$$6 \cdot x_1 - x_2 + 3 \cdot x_3 - s_1 = 5$$

$$6 \cdot x_1 - x_2 + 3 \cdot x_3 + x_4 = 8$$

$$s_1 \geq 0, x_4 \geq 0$$

Let us replace the first constraint by a linear combination of the first and second constraint given by (constraint 2 - constraint 1):

$$x_4 + s_1 = 3$$

Then we obtained an equivalent system of inequalities given by:

$$x_4 + s_1 = 3$$

$$6 \cdot x_1 - x_2 + 3 \cdot x_3 + x_4 = 8$$

$$s_1 \geq 0, x_4 \geq 0$$

Given that s_1 only appear in the first equation and is nonnegative, it can be consider as a slack variable and we can obtain an equivalent system of inequalities given by:

$$x_4 \leq 3$$

$$6 \cdot x_1 - x_2 + 3 \cdot x_3 + x_4 = 8$$

$$x_4 \geq 0$$

b. Indicate how the general interval linear program

$$\begin{aligned}&\max z = \textstyle\sum_{j=1}^{n} c_j \cdot x_j\\ &\text{s.t}\\ &b'_i \le \textstyle\sum_{j=1}^{n} a_{ij} \cdot x_j \le b_i \quad (i = 1, \dots, m)\\ &x_j \ge 0 \quad (i = 1, \dots, m)\end{aligned}$$

Can be formulated as a bounded-variable linear program with m equality constraints. Following the same order of ideas than in part a. we have that for all i the interval inequality can be replaced by the equivalent set of inequalities:

$$\sum_{j=1}^{n} a_{ij} \cdot x_j + y_i \le b_i$$

$$y_i \le b_i - b'_i$$

1. Consider the following LP programming problem with a single constraint:

$$\begin{aligned}&\min \textstyle\sum_{i=1}^{n} c_i \cdot x_i\\ &\text{s.t}\\ &\textstyle\sum_{i=1}^{n} a_i \cdot x_i = b\\ &x_i \ge 0 \quad i = 1, \dots, n\end{aligned}$$

a. Derive a simple test for checking the feasibility of this problem.
   - If $b > 0$ then if there is at least one i' such that $a_{i'} > 0$ then the problem is feasible. The solution $x_{i'} = \frac{b}{a_i}$ and $x_i = 0$ for all $i \neq i'$.
   - If $b < 0$ then if there is at least one i' such that $a_{i'} < 0$ then the problem is feasible. The solution $x_{i'} = \frac{b}{a_i}$ and $x_i = 0$ for all $i \neq i'$.
   - If $b = 0$, the problem is always feasible independent of the value of the $a'_i s$. Solution $x_i = 0$ is a feasible solution for all $i = 1, \dots, n$.

b. Assuming that the optimal cost is finite, develop a simple method for obtaining an optimal solution directly.

Optimal solution is given by:

$$z^* = \min_{\{i | a_i \cdot b > 0\}} \left\{ \frac{b \cdot c_i}{a_i} \right\}$$

Let us denote as i^* an i for which $\frac{b \cdot c_i}{a_i}$ is equal to z^*, then $x_{i^*} = \frac{b}{a_i}$ and $x_i = 0$ for all i such that $i \neq i^*$.

2. Solve the following LP using excel:

$$\begin{aligned}&\max z = 9 \cdot x_2 + x_3 - 2 \cdot x_5 - x_6\\ &\text{s.t}\\ &5 \cdot x_2 + 5 \cdot x_3 + x_4 + \cdot x_5 = 10\\ &x_1 - 15 \cdot x_2 + 2 \cdot x_3 = 2\\ &x_2 + x_3 + x_5 + x_6 = 6\\ &x_j \ge 0 \quad (j = 1, \dots, 6)\end{aligned}$$

Figure 1: Problem 3