Recitation #11

A vendor can set up his truck in one of two locations A, B each day. His profits on the ith day for location A and B are A_i and B_i respectively. However each time he changes location he incurs a cost of c. Suppose the vendor knows A_i and B_i for $i = 1; 2, \ldots, N$. The vendor wishes to maximize his profit over N days.

a. How will you solve the vendor's problem using dynamic programming? Can this problem be formulated as a shortest path problem in a suitably defined graph?

Let $f_j(A)$ be the optimal profit the vendor can make in days $j, j+1, \ldots, N$, assuming he starts day j in location A. Similarly, let $f_j(B)$ be the optimal profit in days j through N assuming he starts day j in location B. Then,

$$f_{j-1}(A) \ = \ A_{j-1} + \max\{f_j(A), f_j(B) - c\} \tag{1}$$

and

$$f_{j-1}(B) \ = \ B_{j-1} + \max\{f_j(A) - c, f_j(B)\} \tag{2}$$

For the network modeling question, we can define 2 nodes A^i and B^i for each day i. We can have edges (A^i, A^{i+1}), (A^i, B^{i+1}), (B^i, A^{i+1}), and (B^i, B^{i+1}), with the corresponding "weights" A_i, $A_i - c$, $B_i - c$, and B_i. Our goal is to find the longest path from the "origin nodes" $\{A^1, B^1\}$ to the "destination nodes" $\{A^N, B^N\}$. The longest of these four longest paths is the optimal profit and the edges in the path give us an optimal policy. (We can negate all the weights and look for a shortest path!).

b. Suppose he worked at location A on day i-1. Let $D = B_i - A_i$. It seems that if D is sufficiently large, it will be optimal to switch to B; and if D is sufficiently small, it will be optimal to continue at A. Can you formalize this? That is, find conditions (in terms of D) under which it is optimal for the vendor to stay at A on day i, and conditions under which it is optimal for her to switch to location B on day i.

For c, it is clear that if $D < 0$ there is no need to switch (because switching would incur a cost). One may think that if $D > c$ it always pays to switch. But that is not true. Suppose $c = 100$, and suppose $A_i = 0, B_i = 101$, and $A_{i+1} = 101, B_{i+1} = 0$. Since $B_i - A_i$ exceeds c, the rule suggests we should switch to B on day i; similarly, we should switch to A on day $i+1$. The associated net benefit is just \$2 because we pay \$200 of the \$202 we earn as switching penalties. If we stay put at A, however, we make a profit of \$101. However, it is easy to see that if $D_i > 2c$, we should switch. This is because the additional profit we make by switching is enough to offset the cost of switching to the other location and switching back.