IEOR E4004: Introduction to Operations Research: Deterministic Models

Jay Sethuraman

HW 9 Solutions

1.

(a) Let's introduce for each node $i \in V$ the binary variables: r_i, b_i, y_i. These indicate if node i is colored red, blue, or yellow, respectively. We now formulate the following IP:

$$\begin{aligned}
&\min \quad 0 \\
&\text{subject to:} \\
&\qquad r_i + b_i + y_i = 1, \forall i \in V \\
&\qquad r_i + r_j \leq 1, \forall (i,j) \in E \\
&\qquad b_i + b_j \leq 1, \forall (i,j) \in E \\
&\qquad y_i + y_j \leq 1, \forall (i,j) \in E \\
&\qquad r_i, b_i, y_i \in \{0,1\}, \forall i \in V
\end{aligned} \tag{1}$$

The first constraint ensures that each node receives exactly one color. The second set of constraints ensures that no two adjacent nodes are both colored red, blue, or yellow.

This LP is feasible if and only if there exists a way to color the nodes of the graph red, blue, and yellow so that no two adjacent nodes have the same color.

(b) Let's introduce the variable x_i for each node $i \in V$. We formulate the following IP:

$$\begin{aligned}
&\max \quad \textstyle\sum_i x_i \\
&\text{subject to:} \\
&\qquad x_i + x_j \leq 1, \forall (i,j) \notin E \\
&\qquad x_i \in \{0,1\}, \forall i \in V
\end{aligned} \tag{2}$$

If a vertex i is assigned a value of $x_i = 1$, then this implies that all the nodes j which are **not** adjacent to it will be assigned the value $x_j = 0$. Thus a feasible 0-1 solution $x = (x_1, x_2, ..., x_n)$ corresponds to set of nodes that are fully connected (the ones with $x_i = 1$), hence a clique. The objective function counts the number of the nodes in this clique. Since we are maximizing over all feasible x, we obtain the maximum size clique.