Lec. 9: Duality Theory wrap-up

Note Title 10/6/2005

Max $\sum c_j x_j$	Min $\sum_i y_i b_i$
$\sum_j a_{ij} x_j \leq b_i$	$\sum_i a_{ij} y_i \geq c_j$
$x_j \geq 0$	$y_i \geq 0$

[Symmetric form]

* Any LP can be converted to this form
* Duality discussion becomes transparent in this setting.
* All results will be developed for this form

Weak duality Thm:

If x is primal feasible & y is dual feasible, then

$$z_P(x) \le z_D(y)$$

Proof:

$$\sum_j c_j x_j \le \sum_j \Big(\sum_i a_{ij} y_i\Big) x_j$$

(Dual feasibility of y, $x \geq 0$)

$$\sum_j \Big(\sum_i a_{ij} y_i\Big) x_j = \sum_i \Big(\sum_j a_{ij} x_j\Big) y_i$$

$$\sum_i \Big(\sum_j a_{ij} x_j\Big) y_i < \sum b_i y_i$$

(Primal feasibility of x, $y \geq 0$)

Strong Duality Thm:

If the Primal problem has an Optimal Solution x^*, then so does the dual (y^*). Moreover,

$$Z_P(x^*) = Z_D(y^*)$$

Note: For strong duality to hold,

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*$$

$$\Rightarrow \quad \sum_j c_j x_j^* = \sum_j \left(\sum_i a_{ij} y_i^*\right) x_j^*$$

$$\& \quad \sum_i b_i y_i^* = \sum_i \left(\sum_j a_{ij} x_j^*\right) y_i^*$$

That is:

$$\left[\begin{array}{lll} x_j^* = 0 & \underline{\text{or}} & \sum_i a_{ij} y_i^* = c_j \\ y_i^* = 0 & \text{or} & \sum_j a_{ij} x_j^* = b_i \end{array}\right]$$

Complementary Slackness Condition.
(CS)

Observe:

CS Conditions $\Rightarrow Z_p(x^*) = Z_D(y^*)$

$Z_p(x^*) = Z_D(y^*) \Rightarrow$ CS Conditions

$$\text{Max} \sum_j c_j x_j \qquad \Bigg| \qquad \text{Min} \sum b_i y_i$$

$$\sum_j a_{ij} x_j \le b_i \qquad \Bigg| \qquad \sum_i a_{ij} y_i \geq c_j$$

$$x_j \geq 0 \qquad \Bigg| \qquad y_i \geq 0$$

$$\equiv \qquad\qquad \equiv$$

$$\text{Max} \sum_j c_j x_j \qquad\qquad \text{Min} \sum_i b_i y_i$$

$$\sum a_{ij} x_j + S_i = b_i \qquad\qquad \sum a_{ij} y_i - w_j = c_j$$

$$x_j \geq 0, \; S_i \geq 0 \qquad\qquad y_i \geq 0, \; w_j \geq 0$$

<u>CS Conditions:</u>

$x_j^* = 0$ or $w_j^* = 0$ <u>ie</u> $x_j^* w_j^* = 0$

$y_i^* = 0$ or $S_i^* = 0$ <u>ie</u> $y_i^* S_i^* = 0$

Implications

1) If x is Primal feasible &

y is dual feasible &

$Z_P(x) = Z_D(y)$,

x is Primal optimal &

y is dual optimal

[Proof: use weak duality]

2) If x is primal feasible
& y is dual feasible
& CS conditions are satisfied
then x is primal optimal
& y is dual optimal.

[<u>Pf</u>: CS conditions $\Rightarrow$
$Z_P(x) = Z_D(y)$]

3) If Primal problem is Unbounded, dual must be infeasible

[Pf: by weak duality thm., any dual feasible soln. gives an upper bound for Primal, So dual cannot be feasible if Primal is unbounded]

CS Conditions illustrated

$$\text{Max } 4x_1 + x_2 + 3x_3$$
$$x_1 + 4x_2 + s_1 = 1$$
$$3x_1 - x_2 + x_3 + s_2 = 3$$
$$x_1, x_2, x_3 \geq 0$$
$$s_1, s_2 \geq 0$$

$$\text{Min } y_1 + 3y_2$$
$$y_1 + 3y_2 - w_1 = 4$$
$$4y_1 - y_2 - w_2 = 1$$
$$y_2 - w_3 = 3$$
$$y_1, y_2 \geq 0$$
$$w_1, w_2, w_3 \geq 0$$

CS conditions:

- $x_1 + 4x_2 = 1$ or $y_1 = 0$ (or both)
- $3x_1 - x_2 + x_3 = 3$ or $y_2 = 0$ (or both)
- $y_1 + 3y_2 = 4$ or $x_1 = 0$
- $4y_1 - y_2 = 1$ or $x_2 = 0$
- $y_2 = 3$ or $x_3 = 0$

[If x & y are optimal for Primal & dual they satisfy CS conditions]

Sometimes CS conditions can be used to validate a "guess".

Prev. Lecture: we saw $y_1 = 1, y_2 = 3$ is feasible for dual

Q: Is $y_1 = 1, y_2 = 3$ optimal?

Using CS conditions:

$y_1 + 3y_2 = 10 \neq 4 \Rightarrow x_1 = 0$

$y_1 \neq 0 \Rightarrow x_1 + 4x_2 = 1 \Rightarrow x_2 = 1/4$

$y_2 \neq 0 \Rightarrow 3x_1 - x_2 + x_3 = 3 \Rightarrow x_3 = 13/4$

$x_1 = 0,\ x_2 = \frac{1}{4},\ x_3 = \frac{13}{4}$ is feasible & so optimal for Primal

& $y_1 = 1, y_2 = 3$ is optimal for dual

(why?)

This trick may not always work.

Max $10x_1 + x_2 + 3x_3$	Min $y_1 + 3y_2$
$x_1 + 4x_2 \leq 1$	$y_1 + 3y_2 \geq 10$
$3x_1 - x_2 + x_3 \leq 3$	$4y_1 - y_2 \geq 1$
$x_1, x_2, x_3 \geq 0$	$y_2 \geq 3$
	$y_1, y_2 \geq 0$

Observe: (1) $y_1 = 1, y_2 = 3$ is still dual optimal [Why?]

(2) $x_1 = 0, x_2 = \frac{1}{4}, x_3 = \frac{13}{4}$ is still Primal optimal [Why?]

(3) CS Conditions + $y_1 = 1, y_2 = 3$ don't give a unique solution [Check]

Max $4x_1 + x_2 + 3x_3$

$x_1 + 4x_2 \le 1$

$3x_1 - x_2 + x_3 \le 3$

$x_1, x_2, x_3 \ge 0$

Min $y_1 + 3y_2$

$y_1 + 3y_2 \ge 4$

$4y_1 - y_2 \ge 1$

$y_2 \ge 3$

$y_1, y_2 \ge 0$

$\equiv$

Max $-y_1 - 3y_2$

$-y_1 - 3y_2 \le -4$

$-4y_1 + y_2 \le -1$

$-y_2 \le -3$

$y_1, y_2 \ge 0$

<u>Initial dictionary</u>

$$
\begin{aligned}
S_1 &= 1 - x_1 - 4x_2 \\
S_2 &= 3 - 3x_1 + x_2 - x_3 \\
\hline
Z_p &= \qquad 4x_1 + x_2 + 3x_3
\end{aligned}
$$

$$
\begin{aligned}
w_1 &= -4 + y_1 + 3y_2 \\
w_2 &= -1 + 4y_1 - y_2 \\
w_3 &= -3 \qquad\quad + y_2 \\
\hline
Z_D &= -y_1 - 3y_2
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= 1 - x_1 - 4x_2 \\
s_2 &= 3 - 3x_1 + x_2 - x_3 \\
\hline
z_p &= 4x_1 + x_2 + 3x_3
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= 1 - x_1 - 4x_2 \\
x_3 &= 3 - 3x_1 + x_2 - s_2 \\
\hline
z_p &= 9 - 5x_1 + 4x_2 - 3s_2
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \frac{1}{4} - \frac{1}{4}x_1 - \frac{1}{4}s_1 \\
x_3 &= \frac{13}{4} - \frac{13}{4}x_1 - \frac{1}{4}s_1 - s_2 \\
\hline
z_p &= 10 - 6x_1 - s_1 - 3s_2
\end{aligned}
$$

Always feasible

work towards optimality

$$
\begin{aligned}
w_1 &= -4 + y_1 + 3y_2 \\
w_2 &= -1 + 4y_1 - y_2 \\
w_3 &= -3 + y_2 \\
\hline
z_D &= -y_1 - 3y_2
\end{aligned}
$$

$$
\begin{aligned}
w_1 &= 5 + y_1 + 3w_3 \\
w_2 &= -4 + 4y_1 - w_3 \\
y_2 &= 3 + w_3 \\
\hline
z_D &= -9 - y_1 - 3w_3
\end{aligned}
$$

$$
\begin{aligned}
w_1 &= 6 + \frac{1}{4}w_2 + \frac{13}{4}w_3 \\
y_1 &= 1 + \frac{1}{4}w_2 + \frac{1}{4}w_3 \\
y_2 &= 3 + w_3 \\
\hline
z_D &= -10 - \frac{1}{4}w_2 - \frac{13}{4}w_3
\end{aligned}
$$

Always satisfies optimality conditions

work towards feasibility

Observations:

Negative transpose of Primal dictionary gives dual dictionary

⇓

$$\boxed{\begin{array}{l}\text{Last row of Primal dictionary} \\ = \text{first column of dual dictionary} \\ = \text{dual optimal solution}\end{array}}$$

* Clue to Proving Strong duality
* Dual Simplex method.