Lec 11: Strong duality Thm; Sensitivity

Note Title 10/14/2005

$$\text{Max } 2x_1 + 3x_2$$

$$-x_1 + x_2 \le 5$$

$$x_1 + 3x_2 \le 35$$

$$x_1 \le 20$$

$$x_1, x_2 \ge 0$$

$$\boxed{x_1 = 20} - s_3$$

$$\boxed{x_2 = 5} - \frac{1}{3}s_2 - \frac{1}{3}s_3$$

$$s_1 = 20 + \frac{1}{3}s_2 - \frac{4}{3}s_3$$

$$z = 55 \underline{- s_2} \underline{- s_3}$$

$$\text{Min } 5y_1 + 35y_2 + 20y_3$$

$$-y_1 + y_2 + y_3 \ge 2$$

$$y_1 + 3y_2 \ge 3$$

$$y_1, y_2, y_3 \ge 0$$

$$\boxed{y_2 = 1} - \frac{1}{3}y_1 + \frac{1}{3}w_2$$

$$\boxed{y_3 = 1} + \frac{4}{3}y_1 + w_1 - \frac{1}{3}w_2$$

$$Z = -55 - 20y_1 \underline{- 20w_1} \underline{- 5w_2}$$

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Primal optimal soln.:

$x_1^* = 20, \quad x_2^* = 5$

From last row of opt. dual dictionary

Coeff. of $w_1 = -20$

$w_2 = -5$

Dual optimal soln:

$y_1^* = 0, \quad y_2^* = 1, \quad y_3^* = 1$

From last row of opt. Primal dictionary:

Coeff. of $s_1 = 0$

$s_2 = -1$

$s_3 = -1$

Observation:

Opt. Soln. to one problem =
negative of the coefficients of
Slack/Surplus variables in the last
row of the optimal dictionary
of the other problem.

Why?

$$Z = 55 - S_2 - S_3 \quad - \circledast$$

We suspect $\{y_2 = 1,\ y_3 = 1,\ y_1 = 0\}$ is an optimal dual soln.

So:

$$Z = 55 - y_2 S_2 - y_3 S_3$$

BUT:

$$Z = 2x_1 + 3x_2$$

$$S_2 = 35 - x_1 - 3x_2$$

$$S_3 = 20 - x_1$$

So:

$$\boxed{2x_1 + 3x_2 = 55 - y_2(35 - x_1 - 3x_2) - y_3(20 - x_1)}$$

As this holds for all x_1, x_2, the coefficients of x_1, x_2 should be the same. & the "constant" term should be zero. So:

$$\left.\begin{aligned} 2 &= y_2 + y_3 \\ 3 &= 3y_2 \end{aligned}\right\} \text{ dual feasibility}$$

$$55 = 35y_2 + 20y_3$$ ← optimal dual soln. [Why?]

→ $y_1 = 0,\ y_2 = 1,\ y_3 = 1$ is dual feasible

→ Obj value = 55, hence dual optimal [why?]

In general:

$$\text{Max} \sum_j c_j x_j \qquad \left[\begin{array}{l} i = 1, 2, \dots m \\ j = 1, 2, \dots n \end{array}\right]$$

$$\sum_j a_{ij} x_j \le b_i$$

$$x_j \geq 0$$

Last row of opt. Primal dictionary:

$$\boxed{Z = Z^* - \sum_j \bar{c}_j x_j - \sum_i \bar{d}_i s_i}$$

By optimality of Primal dictionary

$\bar{c}_j \geq 0$, $\bar{d}_i \geq 0$

for all j for all i

Rewriting $\circledast$ using

$$S_i = b_i - \sum_j a_{ij} x_j$$

$$Z = \sum_j c_j x_j$$

$$\boxed{\begin{aligned} \sum_j c_j x_j &= Z^* - \sum_j \bar{c}_j x_j - \sum_i \bar{d}_i \left\{ b_i - \sum_j a_{ij} x_j \right\} \\ &= Z^* - \sum_j x_j \left\{ \bar{c}_j - \sum_i \bar{d}_i a_{ij} \right\} \\ &\quad - \sum_i b_i \bar{d}_i \end{aligned}}$$

Since this is valid for all $(x_1, x_2, \ldots, x_n)$, we must have:

$$c_j = -\bar{c}_j + \sum_i \bar{d}_i a_{ij} \quad — \text{①}$$

$$Z^* = \sum b_i \bar{d}_i \quad — \text{②}$$

① $\Rightarrow$ $\sum_i a_{ij} \bar{d}_i = c_j + \bar{c}_j \geq c_j$

[because $\bar{c}_j \geq 0$]

Moreover, $\bar{d}_i \geq 0$.

<u>So:</u> $\bar{d}_i$ is dual feasible

By ②, Obj. of dual at this feasible soln. $\{\bar{d}_i\} = Z^*$

<u>So:</u> $\{\bar{d}_i\}$ is dual optimal

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Summary:

We just proved Strong duality thm. for linear Programming:

If an LP has a finite opt. Value, then so does its dual & the opt. Values of the 2 Problems are the Same.

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$$\text{Max } 60x_1 + 30x_2 + 20x_3$$

$$8x_1 + 6x_2 + x_3 + s_1 = 48$$

$$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$$

$$2x_1 + 1.5x_2 + \tfrac{1}{2}x_3 + s_3 = 8$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

$$s_1 = 24 + 2x_2 - 2s_2 + 8s_3$$

$$x_3 = 8 + 2x_2 - 2s_2 + 4s_3$$

$$x_1 = 2 - 1.25x_2 + \tfrac{1}{2}s_2 - 1.5s_3$$

$$z = 280 - 5x_2 - 10s_2 - 10s_3$$

OPTIMAL DICTIONARY

$$x_B = B^{-1}b - B^{-1}N x_N$$

$$z = c_B B^{-1} b - (c_B B^{-1} N - c_N) x_N$$

Feasibility: $B^{-1} b \geq 0$

optimality: $[c_B B^{-1} N - c_N] \geq 0$

Eg: $B = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}$

$$B^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -\frac{1}{2} & -3/2 \end{bmatrix}$$

Changing obj fn. coefficient

- $c_2 = 30$. Suppose $c_2 \uparrow$

  <u>ie.</u> $c_2 = 30 + \delta$

1) For what values of δ is the current basis optimal?

   [Range]

2) Suppose c_2 is changed beyond range. How to reoptimize?

B^{-1}, b remain unchanged

x_2 = non-basic $\Rightarrow$ C_B remains unchanged.

$$\boxed{\text{Only Coefficent of } x_2 \text{ in last row changes.}}$$

$$C_B B^{-1} = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix}$$

$$A_2 = x_2 \text{ column in } A = \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix}$$

$$\bar{c}_2 = C_B B^{-1} A_2 - c_2 = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} - (30+\delta)$$

$$= 35 - (30+\delta) = 5 - \delta$$

Thus:

* c_2 can be decreased by any amount, and
* c_2 can be increased by up to 5

without affecting the optimality of the current basis.

Also, the current soln. does not change.

If c_2 is increased beyond 5, current basis is not optimal.

Use Primal simplex to

continue....