Note Title 10/11/2005

Midterm exam

TUESDAY, Oct. 25

9 - 10:30

→ Open text (Bard, Jensen)

+ class notes

+ hw solutions

→ Calculators allowed

Lecture 10: Dual Simplex, Sensitivity Analysis

Note Title 10/10/2005

$$\text{Max} \quad -5y_1 - 35y_2 - 20y_3$$

$$y_1 - y_2 - y_3 + w_1 = -2$$

$$-y_1 - 3y_2 + w_2 = -3$$

$$y_1, y_2, y_3 \geq 0, \quad w_1, w_2 \geq 0$$

$$w_1 = -2 - y_1 + y_2 + y_3$$

$$w_2 = -3 + y_1 + 3y_2$$

$$z = -5y_1 - 35y_2 - 20y_3$$

→ feasibility conditions are violated

→ Optimality conditions are always satisfied

$$\text{Max} \quad -5y_1 - 35y_2 - 20y_3$$

$$y_1 - y_2 - y_3 + w_1 = -2$$

$$-y_1 - 3y_2 + w_2 = -3$$

$$y_1, y_2, y_3 \geq 0, \quad w_1, w_2 \geq 0$$

$$w_1 = -2 - y_1 + y_2 + y_3$$

$$w_2 = -3 \pm y_1 \pm 3y_2 \Rightarrow \text{Pivot row}$$

$$z = -5y_1 - 35y_2 - 20y_3$$

Entering variable

$$\min\left\{\frac{5}{1}, \frac{35}{3}\right\}$$

y_1 enters

Leaving variable

[Can be any variable with a negative value. Usually most negative]

$$y_1 = 3 - 3y_2 + w_2$$

$\Leftarrow w_1 = -5 + 4y_2 + y_3 - w_2$ (Pivot row)

$y_1 = 3 - 3y_2 + w_2$

$z = -15 - 20y_2 - 20y_3 - 5w_2$

$\Uparrow$

w_2 does not participate in ratio test

Ratio. $= \min\left\{ \frac{20}{4}, \frac{20}{1} \right\}$

$[y_2]$ $[y_3]$

y_2 enters

$4y_2 = 5 - y_3 + w_1 + w_2$

$y_2 = \frac{5}{4} - \frac{1}{4}y_3 + \frac{1}{4}w_1 + \frac{1}{4}w_2$

$$y_2 = \frac{5}{4} - \frac{y_3}{4} + \frac{w_1}{4} + \frac{w_2}{4}$$

$$\Leftarrow y_1 = -\frac{3}{4} + \frac{3y_3}{4} - \frac{3w_1}{4} + \frac{w_2}{4}$$

$$Z = -40 - 15y_3 - 5w_1 - 10w_2$$

y_1 leaves

w_1 does not participate in ratio test

$$\text{Min}\left\{\frac{15}{3/4}, \frac{10}{1/4}\right\}$$

y_3 enters.

What if:

$$y_1 = -3 - 3y_3 - w_1 - w_2$$

No candidate for min. ratio test ?

$$y_1 + 3y_3 + w_1 + w_2 = -3$$

Infeasible

$$y_2 = \frac{5}{4} - \frac{1}{4} - \frac{y_1}{3/4} - \frac{w_1}{4} + \frac{w_2}{1/2}$$

$$y_2 = 1 - \frac{1}{3}y_1 + \frac{w_2}{3}$$

$$y_3 = 1 + \frac{4}{3}y_1 + w_1 - \frac{1}{3}w_2$$

$$Z = -55 - 20y_1 - 20w_1 - 5w_2$$

Optimal dictionary

Observe: "Negative transpose" property

See Pg. 81 of text
(Also class discussion)

Max $2x_1 + 3x_2$

$-x_1 + x_2 \leq 5$

$x_1 + 3x_2 \leq 35$

$x_1 \leq 20$

$x_1, x_2, x_3 \geq 0$

$$x_1 = 20 - s_3$$

$$x_2 = 5 - \frac{1}{3}s_2 + \frac{1}{3}s_3$$

$$s_1 = 20 + \frac{1}{3}s_2 - \frac{4}{3}s_3$$

$$Z = 55 - s_2 - s_3$$

$$y_2 = 1 - \frac{1}{3}y_1 + \frac{w_2}{3}$$

$$y_3 = 1 + \frac{4}{3}y_1 + w_1 - \frac{1}{3}w_2$$

$$z = -55 - 20y_1 - 20w_1 - 5w_2$$

"Complementary" dictionary

y_2, y_3 basic $\Rightarrow$ s_2, s_3 non-basic

y_1, w_1, w_2 non-basic $\Rightarrow$ s_1, x_1, x_2 basic.

y_1 Column $\equiv -s_1$ row, So: $s_1 = 20 + \frac{1}{3}s_2 - \frac{4}{3}s_3$

$$\begin{bmatrix} -1/3 \\ 4/3 \\ -20 \end{bmatrix} \qquad \begin{bmatrix} 20 & \frac{1}{3} & -\frac{4}{3} \end{bmatrix}$$

w_2 Column $\equiv -x_2$ row. So: $x_2 = 5 - \frac{1}{3}s_2 + \frac{1}{3}s_3$

$\begin{bmatrix} 1/3 \\ -1/3 \\ -5 \end{bmatrix}$ $\quad [5 \ -1/3 \ 1/3]$

w_1 Column $\equiv -x_1$ row $\qquad x_1 = 20 - s_3$

$\begin{bmatrix} 0 \\ 1 \\ -20 \end{bmatrix}$ $\quad [20 \ 0 \ -1]$

Col. of Constants $\equiv -$ last row $\qquad z = 55 - s_2 - s_3$

$\begin{bmatrix} 1 \\ 1 \\ -55 \end{bmatrix}$ $\quad [55 \ -1 \ -1]$

Red & Green dictionaries are **Complementary**

Matrix Notation

$$\text{Max} \quad -5y_1 - 35y_2 - 20y_3$$

$$y_1 - y_2 - y_3 + w_1 = -2$$

$$-y_1 - 3y_2 + w_2 = -3$$

$$y_1, y_2, y_3 \geq 0, \quad w_1, w_2 \geq 0$$

$$c = [-5, -35, -20, 0, 0]^T \qquad x = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ w_1 \\ w_2 \end{bmatrix} \qquad b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 \\ -1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{MAX} \quad c^T x$$

$$Ax = b$$

$$x \geq 0.$$

Suppose y_1 & w_1 are basic. How to write the associated dichonary ?

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

(The Columns of A Correcponding to y_1 & w_1)

$$x_B = \begin{bmatrix} y_1 \\ w_1 \end{bmatrix}, \quad x_N = \begin{bmatrix} y_2 \\ y_3 \\ w_2 \end{bmatrix}, \quad N = \begin{bmatrix} -1 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$c_B = \begin{bmatrix} -5 \\ 0 \end{bmatrix}, \quad c_N = \begin{bmatrix} -35 \\ -20 \\ 0 \end{bmatrix}$$

The given linear Program is:

$$\text{Max} \quad c_B^T x_B + c_N^T x_N$$

$$B x_B + N x_N = b$$

$$x_B, x_N \geq 0$$

B is an invertible matrix (Why?)

Dictionary:

$$B x_B + N x_N = b$$

$$\Rightarrow B x_B = b - N x_N$$

$$\Rightarrow B^{-1} B x_B = B^{-1} b - B^{-1} N x_N$$

$$\Rightarrow \boxed{x_B = B^{-1} b - B^{-1} N x_N}$$

first m rows

$$Z = c_B x_B + c_N x_N$$

$$\Rightarrow \boxed{Z = c_B B^{-1} b + (c_N - c_B B^{-1} N) x_N}$$

Last row

If $\{y_1, w_1\}$ are the basic variables:

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

So: $$B^{-1}b = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$B^{-1}NX_N = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_2 \\ y_3 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -y_2 - y_3 \\ -3y_2 + w_2 \end{bmatrix} = \begin{bmatrix} 3y_2 - w_2 \\ -4y_2 - y_3 + w_2 \end{bmatrix}$$

So: the dictionary for basis $\{y_1, w_1\}$ is (except for last row)

$$\begin{bmatrix} y_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} - \begin{bmatrix} 3y_2 - w_2 \\ -4y_2 - y_3 + w_2 \end{bmatrix}$$

or

$$y_1 = 3 - 3y_2 + w_2$$
$$w_1 = -5 + 4y_2 + y_3 - w_2$$

Last row:

$$Z = C_B B^{-1} b + (C_N - C_B B^{-1} N) x_N$$

$$C_B B^{-1} b = \begin{bmatrix} -5, 0 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = -15$$

$$C_B B^{-1} N = \begin{bmatrix} -5 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -15 & 0 & 5 \end{bmatrix}$$

$$(C_N - C_B B^{-1} N) = (-35, -20, 0) - (-15, 0, 5)$$

$$= (-20, -20, -5)$$

So: last row reads:

$$z = -15 - 20y_2 - 20y_3 - 5w_2$$

Dictionary:

$$y_1 = 3 - 3y_2 + w_2$$

$$w_1 = -5 + 4y_2 + y_3 - w_2$$

$$z = -15 - 20y_2 - 20y_3 - 5w_2$$

We saw this earlier!

Summary:

* Given any basis, we can recover the dictionary associated with the basis as long as we know B^{-1}
* basis of revised Simplex method.