IEOR E4004: Introduction to Operations Research: Deterministic Models

Jay Sethuraman

HW 9 Solutions

1. This is simply the binary knapsack problem discussed in class. We can draw a table with 20 rows (one each for capacities 0 through 19), and 5 columns (one for each item). The entry (i, j) in this table would correspond to the best value achievable with a knapsack of capacity i using items 1 through j. We can first determine the first column; then use the knapsack recursion to determine the second column, etc. The entry we want is the $(19, 5)$ entry on the table. The complete table is not shown in the solutions. The answer to the problem is 38, and is obtained by including items 1, 2, and 5.

2. Let $d^k(j)$ be the shortest path length from node s to j when just nodes $s, \ldots, k$ can be used. We can assume WLOG that nodes are numerated from 1 to n, in which $s = 1$ and $t = n$. We are interested in compute $d^n(n)$. We can also assume WLOG that if $(k, j) \notin A$ then we assume that $c_{kj} = \infty$.

   Let us define the initial step as:

$$d^1(1) = 0$$
$$d^1(v) = \infty \forall v \in N \setminus \{1\}$$

   And the DP recursive step as:

$$d^k(j) = \min\left\{d^{k-1}(j), \max_{j \in N, j \neq k}\left\{d^{k-1}(k), c_{kj}\right\}\right\}$$

3. Let y_i be the inventory level at the beginning of day i. This will be our state variable. Then, given that x_k is the amount we produce on day i and d_i is the demand that has to meet on day k, we can note the following state dynamics:

$$y_0 = 0 \tag{1}$$

$$y_{i+1} = y_i + x_i - d_i \tag{2}$$

   Let $1\{\cdot\} = 1$ if the condition inside the brackets is met and 0 otherwise. The dynamic programming recursion will then be:

$$J_N(y_N) = 1\{y_N < d_N\}(K_N + c_N(d_N - y_N)) \tag{3}$$

$$J_i(y_i) = \min_{x_i : x_i + y_i \geq d_i} \{K_i 1\{x_i > 0\} + c_i x_i + h_i(y_i + x_i - d_i) + J_{i+1}(y_{k+1})\} \quad (4)$$

4. Let $d(i, y_j)$ be the maximum profit obtained when just defects $1, \ldots, i$ can be cut and length of roll is y_j. We aimed to compute:

$$d(N-1, y_N)$$

We may assume that $j > i$.
The initial step is:

$$d\left(0, y_i\right) = v\left(y_i, 0\right) \forall i = 1, \ldots, N$$

The DP recursive step is given by:

$$d\left(i, y_j\right) = \max\left\{d\left(i-1, y_i\right) + v\left(y_j - y_i, j-i-1\right), d\left(i-1, y_j\right)\right\}$$

Backtracking how we arrives to $d(N-1, y_N)$ we will find through which defects we should cut the roll.