Homework #5

Problem 4.7 a. We have $N = 2, T = 3$. Let item 1 be the boots, and 2 be the shoes. so $v_1 = 10, v_2 = 5, c_1 = 2, c_2 = 1, r = 3, o = 6, k_1 = 1, k_2 = 0.5$. The primal is given by

$$\begin{array}{lll}
\min z = & \sum_{i=1}^{2}\sum_{t=1}^{3}(v_i X_{it} + c_i I_{it}) + \sum_{t=1}^{3}(rW_t + oO_t) & \\
\text{s.t.} & & \text{Dual var.} \\
& X_{it} + I_{i,t-1} - I_{it} = d_{it}\ \forall\ i,t & y_{it} \\
& \sum_i k_i X_{it} - W_t - O_t = 0\ \forall\ t & a_t \\
& -W_t \geq -2000\ \forall\ t & b_t \\
& 0.25W_t - O_t \geq 0\ \forall\ t & e_t \\
& X_{it}, I_{i,t}, W_t \geq 0\ \forall\ i,t &
\end{array}$$

b. Dual problem is

$$\begin{array}{ll}
\max\ w & \sum_{i=1}^{2}\sum_{t=1}^{3} d_i t y_{it} - \sum_{t=1}^{3} 2000 b_t \\
\text{s.t.} & y_{it} + k_i a_t \leq v_i\ \forall\ i,t \\
& -y_{it} + y_{i,t+1} \leq c_i\ \forall\ i, t = 1,2 \\
& -y_{i3} \leq c_i\ \forall\ i \\
& -a_t - b_t + 0.25e_t \leq r\ \forall\ t \\
& -a_t - e_t = o\ \forall\ t \\
& b_t, e_t \geq 0, y_{it}, a_t \text{ urs}
\end{array}$$

c. (Refer to Section 4.8 Mathematical Economics in the textbook for more detailed explanation on this topic.) Dual variables are the prices of the resources and the products of the company set by the market (or another company). Market wants to set these prices to maximize its profit (or minimize the profit of the company). The markets profit comes from selling products of the company. The variables y_{it} are the selling prices of these products. If the company does not use its resources fully, it can sell them to obtain profit. Therefore, this is the cost of the market. We can write the profit of the market as follows

$$\sum_{i=1}^{2}\sum_{t=1}^{3} d_i t y_{it} - \sum_{t=1}^{3} 2000 b_t + \sum_{i,t}(-y_{it} - k_i a_t + v_i)X_{i,t} + \sum_{i,t}(+y_{it} - y_{i,t+1} + c_i)I_{i,t}$$
$$+ \sum_t (a_t + b_t - 0.25e_t + r)W_t + \sum_t (a_t + e_t + o)O_t$$

The constraints of the dual can be interpreted as follows. If $-y_{it} - k_i a_t + v_i < 0$, then the company will sell large amount of products, ie will increase X_{it} which will decrease the profit of the market. Therefore the market will set prices such that $-y_{it} - k_i a_t + v_i \geq 0$. Other constraints can also be interpreted in the same way.

Problem 4.12 a. An additional dollar at the end of year 1 would be invested in project C which will return a shadow price of 1.25 (remember that the shadow price give us the amount that the objective value will increase if we have an additional unit in the RHS).
On the other hand if we have an extra dollar at the end of year 0 we can invested in project A and then invest the 0.6 dollars that we obtain from project A invest them in project C to produce an extra income of $0.6 \cdot 1.25 + 0.6 = 1.35$ at the end of period 2, which by definition of shadow price is the shadow price associated with the constraint of end of period 0.

b. The discount factor for time i is defined to be the present value (time $i = 0$) of one dollar received at time i. From exercise 8 we have that discount factor for time i int he horizon-value model is given by:

$$\rho_i = \frac{y_i^*}{y_0^*} \tag{1}$$

Under the assumption that $y_0^* > 0$.

$$\rho_0 = \tfrac{1.35}{1.35} = 1$$
$$\rho_1 = \tfrac{1.25}{1.35} = 0.926$$
$$\rho_2 = \tfrac{1}{1.35} = 0.74074$$

c. The discounted present value for each project:

$$\textstyle\sum_{i=0}^{2} \rho_i \cdot c_{iA} = 1 \cdot \rho_0 - 0.6 \cdot \rho_1 - 0.6 \cdot \rho_2 = 0$$
$$\textstyle\sum_{i=0}^{2} \rho_i \cdot c_{iB} = 1 \cdot \rho_0 - 1.1 \cdot \rho_1 = -0.01852$$
$$\textstyle\sum_{i=0}^{2} \rho_i \cdot c_{iC} = 1 \cdot \rho_1 - 1.25 \cdot \rho_2 = 0$$
$$\textstyle\sum_{i=0}^{2} \rho_i \cdot c_{iD} = 1 \cdot \rho_0 - 1.3 \cdot \rho_2 = 0.037037$$

They have the opposite sign than the corresponding reduce cost. Be careful when you analyze project B, don't forget than it have upper bound and given that the optimal solution is at the upper bound and is a maximization problem we have that the reduce cost could be positive at optimality.

d. The discounted present value of project E and F are -0.10185 and -0.09259 respectively. By discussion in part c. (about the sign of the reduce cost) they are attractive for investment. If funds are transfered to project E, F is not promising because E has a more negative discounted present value. Yes the discount factors will change.

Problem 4.15 a. Adding a new constraint to an LP (maximization) the optimal objective value will not increase. Therefore, if the current solution satisfies the new constraint, then it is optimal

b. After adding the constraint and pivoting we get the following tableau which is in dual canonical form.

BV	value	x_1	x_2	x_3	x_4	x_5	x_6
x_1	36	1	6	0	4	-1	0
x_3	6	0	-1	1	- 1	1/2	0
x_6	-12	0	-4	0	-3	3/2	1
$-z$	-294	0	-9	0	-11	-1/2	0

c. Apply the dual simplex method to get the final tableau (x_6 leaves and x_2 enters)

BV	value	x_1	x_2	x_3	x_4	x_5	x_6
x_1	18	1	0	0	-1/2	-1/4	3/2
x_3	9	0	0	1	- 1/4	3/8	-1/4
x_2	3	0	1	0	3/4	-1/8	-1/4
$-z$	-267	0	0	0	-17/4	-13/8	-9/4

d. Yes, using the dual simplex method we can use the current solution of the original LP solve the new LP.

Problem 4.21 Player 1 solves

$$\begin{array}{ll} \max & w \\ \text{s.t.} & x_1 + 5x_2 - w \geq 0 \\ & 4x_1 + 2x_2 - w \geq 0 \\ & 3x_1 + 3x_2 - w \geq 0 \\ & x_1 + x_2 = 1 \\ & x_i \geq 0 \end{array} \tag{2}$$

and the optimal solution to 2 is $x_1^* = 0.5$, $x_2^* = 0.5$ and $w^* = 3$ and player 2 solves

$$\begin{array}{ll} \min & v \\ \text{s.t.} & y_1 + 4y_2 + 3y_3 - v \leq 0 \\ & 5y_1 + 2y_2 + 3y_3 - v \leq 0 \\ & y_1 + y_2 + y_3 = 1 \\ & y_i \geq 0 \end{array} \tag{3}$$

Since problem (3) is the dual of problem (2), using complementary slackness conditions one can solve problem 3 and see that it does not have unique solution.

problem 4.25 a.

$$\begin{aligned} \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} y_i \cdot a_{ij} \cdot x_j &= \textstyle\sum_{i=1}^{n} \left[\sum_{j=1}^{n} x_i \cdot a_{ij} \cdot x_j \right] \\ &= \textstyle\sum_{i=1}^{n} \sum_{j=i}^{n} [x_i \cdot a_{ij} \cdot x_j + x_j \cdot a_{ji} \cdot x_i] \\ &= \textstyle\sum_{i=1}^{n} \sum_{j=i}^{n} [x_i \cdot a_{ij} \cdot x_j + x_j \cdot -a_{ij} \cdot x_i] \\ &= 0 \end{aligned} \tag{4}$$

b. Following from part (a) the LHS sums to:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} y_i \cdot a_{ij} \cdot x_i = 0 \tag{5}$$

And the RHS sums to:

$$w \cdot \left(\sum_{i=1}^{n} x_i \right) = w \tag{6}$$

Then we have:

$$w \leq 0 \tag{7}$$

Repeat the same argument for the dual (the game of the other player), you will obtian:

$$v \geq 0 \tag{8}$$

By weak duality it follows that:

$$v = w = 0 \tag{9}$$

problem 4.26 a. Yes, the problem is skew symmetric. To check if the problem is skew symmetric we just need to check if the given matrix satisfies $a_{ij} = -a_{ji}$ and if the number of rows is equal to the number of columns which clearly is holding in this case.

b. First we need to define what does it means to solve the game. To solve the game we just need to check that:

$$\sum_{i=1}^{n} \bar{x}_i + \sum_{i=1}^{n} \bar{y}_i + t = 1 \tag{10}$$

$$\begin{aligned} \textstyle\sum_{i=1}^{n} \bar{x}_i + \sum_{i=1}^{n} \bar{y}_i + t &= t \cdot [\textstyle\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i + 1] \\ &= \frac{1}{(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i + 1)} \cdot [\textstyle\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i + 1] \\ &= 1 \end{aligned} \tag{11}$$

Because v and w can be made arbitrarily large and small respectively to guarantee that the other constraint of the game are holding.

c. Checking that it solves the primal:

$$\begin{aligned} \textstyle\sum_{j=1}^{n} a_{ij} \cdot x_j - b_i &\leq \textstyle\sum_{j=1}^{n} a_{ij} \cdot \frac{\bar{x}_j}{t} - b_i \\ &= \frac{1}{t} \cdot \textstyle\sum_{j=1}^{n} a_{ij} \cdot \bar{x}_j - b_i \\ &= \frac{1}{t} \cdot \left[\textstyle\sum_{j=1}^{n} a_{ij} \cdot \bar{x}_j - b_i \cdot t\right] \\ &\leq 0 \end{aligned} \tag{12}$$

Under the assumption that $t > 0$.

Checking that it solves the dual:

$$\begin{aligned} \textstyle\sum_{i=1}^{n} y_i \cdot a_{ij} - c_j &= \frac{1}{t} \cdot \textstyle\sum_{i=1}^{n} \bar{y}_i \cdot a_{ij} - c_j \\ &= \frac{1}{t} \cdot [\textstyle\sum_{i=1}^{n} \bar{y}_i \cdot a_{ij} - c_j \cdot t] \\ &= -\frac{1}{t} \cdot [c_j \cdot t - \textstyle\sum_{i=1}^{n} \bar{y}_i \cdot a_{ij}] \\ &\geq 0 \end{aligned} \tag{13}$$