Homework #3

Problem 1. (a) Let x_k be the number of officers assigned to highway k, $k = 1, 2, \ldots, K$. The following linear program

$$\begin{array}{ll} \max & \sum_{k=1}^{K} x_k r_k \\ s.t. & l_k \leq x_k \leq u_k, \text{for k} = 1, 2, \ldots, K \\ & \sum_k x_k \leq D \\ & x_k \geq 0 \end{array} \qquad (1)$$

The first constraint ensures that the number of officers assigned to each highway is between the stated lower and upper bounds. The second constraint ensures that we do not assign more officers than are available. The objective function represents the sum of all the individual highway reductions.

(b) In this part the variables and the constraints remain the same, but the objective function now becomes:

$$\max \min\{r_1 x_1, r_2 x_2, \ldots, r_K x_K\},$$

where the maximization is over all possible choices of $(x_1, x_2, \ldots, x_K)$ satisfying the constraints of the problem. As written this is a nonlinear objective. Fortunately, we can rewrite it as a linear objective function by introducing a new variable z which represents

$$\min\{r_1 x_1, r_2 x_2, \ldots, r_K x_K\}.$$

Thus the mathematical program becomes:

$$\begin{array}{ll} \max & z \\ s.t. & z = \min\{r_1 x_1, r_2 x_2, \ldots, r_K x_K\} \\ & l_k \leq x_k \leq u_k, \text{for k} = 1, 2, \ldots, K \\ & \sum_k x_k \leq D \\ & x_k \geq 0 \end{array} \qquad (2)$$

Obviously, this is not an LP because the first constraint is nonlinear. But by replacing the first constraint by

$$z \leq \min\{r_1 x_1, r_2 x_2, \ldots, r_K x_K\},$$

we can recover a linear program, because this new constraint can be rewritten as K linear constraints:

$$z \leq r_k x_k \quad k = 1, 2, \ldots, K.$$

Observe that changing

$$z = \min\{r_1 x_1, r_2 x_2, \ldots, r_K x_K\}$$

to

$$z \leq \min\{r_1 x_1, r_2 x_2, \ldots, r_K x_K\}$$

is harmless because we are *maximizing* z and so even if we "allow" values of z that are smaller than $\min\{r_1x_1, r_2x_2, \ldots, r_Kx_K\}$, an optimal solution will always have $z = \min\{r_1x_1, r_2x_2, \ldots, r_Kx_K\}$. Thus, the given problem can be formulated as the following LP:

$$\begin{array}{ll} \max & z \\ s.t. & l_k \leq x_k \leq u_k, \text{for all k} \\ & \sum_k x_k \leq D \\ & z \leq r_k x_k, \text{for all k} \\ & x_k \geq 0 \end{array} \tag{3}$$

At optimality, z will be exactly equal to the greatest possible minimum reduction over all feasible assignments.

Note that a more complicated way of solving this problem involves introducing variables x_{ij} to indicate if officer i is assigned to highway j. This is not necessary since any two officers reduce traffic on any given highway by the same amount. So, we do not care which man is assigned to which highway. All we care about is how *many* officers are assigned to each highway.

problem 2. From the given optimal dictionary we can observed that x_4 is a nonbasic variable with 0 coefficient in the last row (the row that correspond to the objective function) in the optimal dictionary. Then from the point of view of optimality x_4 can take any value without affecting the optimal value. Now we need to check for which values of x_4 we will still keep feasibility, in other words, all the variables need to be nonnegative.
From the first row of the given dictionary we have that:

$$x_3 = 4 - 5 \cdot x_4 \geq 0 \Rightarrow x_4 \leq \frac{4}{5} \tag{4}$$

From the second row of the given dictionary we have that:

$$x_2 = 2 + 3 \cdot x_4 \geq 0 \Rightarrow x_4 \geq \frac{-2}{3} \tag{5}$$

But given that by non negativity constraint $x_4 \geq 0$, then just row 1 is constraining x_4, in other words if we increase x_4 more than $\frac{4}{5}$ then x_3 will become negative, violating it corresponding non negativity constrain. Then we can conclude that any point of the form:

$$x = (0, 2 + 3 \cdot x_4, 4 - 5 \cdot x_4, x_4, 0, 0) \quad \text{with } 0 \leq x_4 \leq \frac{4}{5} \tag{6}$$

is an optimal solution. An analogous analysis follow for x_1 which is also a nonbasic variable with 0 reduced cost.

problem 3. Let us rewrite the given problem in canonical form:

$$\begin{array}{l} \max z = 2 \cdot x_1 - 3 \cdot x_2 + x_3 - 4 \cdot x_4 \\ \text{s.t} \\ 2 \cdot x_1 - x_2 + 3 \cdot x_3 - 5 \cdot x_4 + x_5 = 20 \\ x_1 + 2 \cdot x_2 - x_3 + 4 \cdot x_4 - x_6 + x_7 = 2 \\ x_4 + x_8 = 20 \\ x_i \geq 0, \quad i = 1, \ldots, 8 \end{array} \tag{7}$$

Then the phase I problem:

$$
\begin{aligned}
&\min z = x_7 \\
&\text{s.t} \\
&2 \cdot x_1 - x_2 + 3 \cdot x_3 - 5 \cdot x_4 + x_5 = 20 \\
&x_1 + 2 \cdot x_2 - x_3 + 4 \cdot x_4 - x_6 + x_7 = 2 \\
&x_4 + x_8 = 20 \\
&x_i \geq 0, \quad i = 1, \ldots, 8
\end{aligned} \tag{8}
$$

Then for phase I we have:

Figure 1: Phase I problem 3

		x1	x2	x3	x4	x5	x6	x7	x8
x5	20	2	-1	3	-5	1	0	0	0
x7	2	1	2	-1	4	0	-1	1	0
x8	20	0	0	0	1	0	0	0	1
z	2	1	2	-1	4	0	-1	0	0

x2 enter the basis and x_7 leave the basis

		x1	x2	x3	x4	x5	x6	x7	x8
x5	21	2.5	0	2.5	-3	1	-0.5	0.5	0
x2	1	0.5	1	-0.5	2	0	-0.5	0.5	0
x8	20	0	0	0	1	0	0	0	1
(-z)	0	0	0	0	0	0	0	-1	0

Then replacing the original objective function, the Phase II initial dictionary is:

		x1	x2	x3	x4	x5	x6	x7	x8
x5	21	2.5	0	2.5	-3	1	-0.5	0.5	0
x2	1	0.5	1	-0.5	2	0	-0.5	0.5	0
x8	20	0	0	0	1	0	0	0	1
(-z)	3	3.5	0	-0.5	2	0	-1.5	1.5	0

problem 4. a.
- $E < 0$ and $C \geq 0$

b.
- $E = 0$ and $C > 0$
- $E = 0$ and $C = 0$ and $D \leq 0$

c.
- $E > 0$ and $C \geq 0$ and $(B \leq 0$ and $D \leq 0)$

d.
- $C < 0$ and $D \geq 0$
- $C < 0$ and $D < 0$ and $\frac{C}{D} > \frac{1}{B}$ and $B > 0$. This arises from the case in which even though C is negative, we can try to increase x_1 such that x_5 becomes positive but any increase in x_1 cause that x_4 becomes negative.

e. If current solution is not optimal $E > 0$ is a necessary condition. Then x_1 is entering the basis and x_4 or(exclusive) x_5 are leaving the basis.

x_1 is entering the basis and x_4 is leaving and x_4 is the only candidate to leave the basis.
This case can happen if $B > 0$ and $D \leq 0$. After applying one pivot operation we have that the objective value increase by $\frac{E}{B}$.

- x_1 is entering the basis and x_4 is leaving the basis and x_4 and x_5 are candidates to leave the basis.
This case can happen if $B > 0$ and $D > 0$ but the ratio test is won by x_4 in other words $\frac{1}{B} = \min\left\{\frac{1}{B}, \frac{C}{D}\right\}$. After applying one pivot operation we have that the objective value increase by $\frac{E}{B}$.

- x_1 is entering the basis and x_5 is leaving the basis and x_5 is the only candidate to leave the basis.
  This case can happen if $B \leq 0$ and $D > 0$. After applying one pivot operation we have that the objective value increase by $\frac{C \cdot E}{D}$.
- x_1 is entering the basis and x_5 is leaving the basis and x_4 and x_5 are candidates to leave the basis.
  This case can happen if $B.0$ and $D > 0$ but the ratio test is won by x_5 in other words $\frac{C}{D} = \min\left\{\frac{1}{B}, \frac{C}{D}\right\}$. After applying one pivot operation we have that the objective value increase by $\frac{C \cdot E}{D}$.

  Remark: Another case which is include under the cases in which both x_4 and x_5 are candidates to leave the basis arise when both variables are tie in the ratio test. In that case any of then can leave the basis and the objective value increase by $\frac{E}{B} = \frac{C \cdot E}{D}$.

problem 5. a. Given that the constants c_j, a_j and u_j for $j = 1, \ldots, n$ are all positive we have that optimal solution to the given LP program is:

$$\begin{aligned} x_1^* &= \min\left\{\frac{b}{a_1}, u_1\right\} \\ x_i^* &= \min\left\{\max\left\{0, \frac{b - \sum_{j=1}^{i} a_j \cdot x_j}{a_i}\right\}, u_i\right\} \quad i = 2, \ldots, n \end{aligned} \tag{9}$$

b. The initial basic feasible solution will be x_1 in the basis and all other variables out of the basis. If the solution is not optimal will try to enter x_2 to the basis and x_1 will leave the basis but it will stay at the upper bound. And so on, until LFH value of the constraint equal to RHS value of the constraint, moment in which we will achieve optimality.

problem 6 Solved in excel. Here we present a printout of the results.

Figure 2: Problem 6