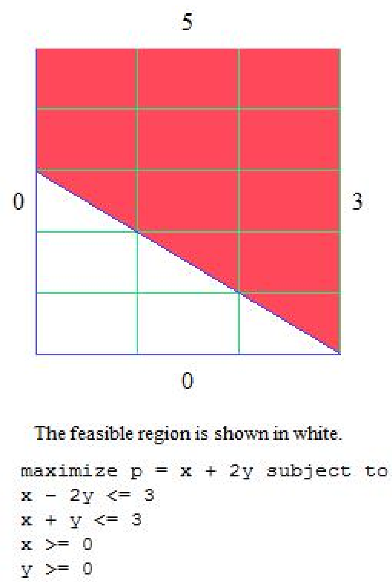
Homework #1

1.1 a.

$$
\begin{aligned}
&\max z = x_1 + 2 \cdot x_2 \\
&\text{s.t} \\
&x_1 - 2 \cdot x_2 \leq 3 \\
&x_1 + x_2 \leq 3 \\
&x_1 \geq 0 \quad x_2 \geq 0
\end{aligned} \tag{1}
$$

Figure 1: Problem 1.a



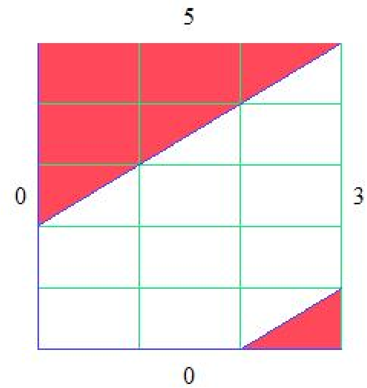
b.

$$
\begin{aligned}
&\min z = x_1 + x_2 \\
&\text{s.t} \\
&x_1 - x_2 \leq 2 \\
&x_1 - x_2 \geq -2 \\
&x_1 \geq 0 \quad x_2 \geq 0
\end{aligned} \tag{2}
$$

c.

$$
\begin{aligned}
&\max z = x_1 + x_2 \\
&\text{s.t} \\
&x_1 - x_2 \leq 2 \\
&x_1 - x_2 \geq -2 \\
&x_1 \geq 0 \quad x_2 \geq 0
\end{aligned} \tag{3}
$$

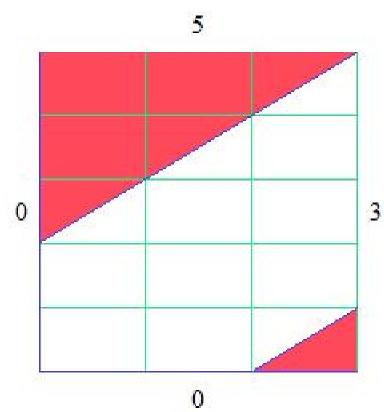
Figure 2: Problem 1.b



The feasible region is shown in white.

Vertex	Lines Through Vertex	Value of Objective
(2,0)	x-y = 2; y = 0	2
(0,2)	x-y = -2; x = 0	2
(0,0)	x = 0; y = 0	0 Minimum

Figure 3: Problem 1.c



The feasible region is shown in white.

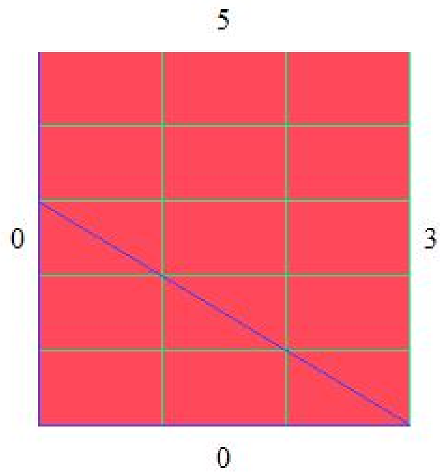
Vertex	Lines Through Vertex	Value of Objective
(2,0)	x-y = 2; y = 0	2
(0,2)	x-y = -2; x = 0	2
(0,0)	x = 0; y = 0	0

***Unbounded Feasible Region -- No Optimal Solution ***

d.

$$\begin{aligned}
&\max z = 3 \cdot x_1 + 4 \cdot x_2 \\
&\text{s.t} \\
&x_1 - 2 \cdot x_2 \geq 4 \\
&x_1 + x_2 \leq 3 \\
&x_1 \geq 0 \quad x_2 \geq 0
\end{aligned} \tag{4}$$

Figure 4: Problem 1.d



The feasible region is shown in white.

Empty Feasible Region -- No Optimal Solution

1.4 a. Let us define the variables x_B, x_W as the total number of produced bottles of bourbon and blended whiskey respectively. From the information in the problem we also know that whatever is produced is sold.

$$\begin{aligned}
&\max(6-3) \cdot x_B + (5.4-2) \cdot x_W \\
&\text{s.t} \\
&3 \cdot x_B + 4 \cdot x_W \leq 20000 && \text{Constraint on machine capacity} \\
&3 \cdot x_B + 2 \cdot x_W \leq 4000 + 45\% \cdot 6 \cdot x_B + 30\% \cdot 5.4 \cdot x_W && \text{Constraint on working capital} \\
&x_B \geq 0 \quad X_W \geq 0
\end{aligned} \tag{5}$$

b. Optimal production mix to schedule The optimal solution is:

$$\begin{aligned}
x_B &= \tfrac{20000}{3} \\
x_W &= 0
\end{aligned} \tag{6}$$

2.12 LP from part a. in standard form:

$$\begin{aligned}
&x_1 - 6 \cdot (x_2^+ - x_2^-) + x_3 - x_4 + x_5 = 5 \\
&-2 \cdot (x_2^+ - x_2^-) + 2 \cdot x_3 - 3 \cdot x_4 - x_6 + x_7 = 3 \\
&-3 \cdot x_1 + 2 \cdot x_3 - 4 \cdot x_4 + x_8 = 1 \\
&x_i \geq 0 \quad i = 1,3,4,5,6,7,8 \quad x_2^+ \geq 0 \quad x_2^- \geq 0
\end{aligned} \tag{7}$$

Note that you can have several equivalent LP formulations for the same problem. So you may have found a different standard form formulation for this problem.

2.4 LP in standard form

$$\begin{aligned} &\max x_1 \\ &-x_1 + x_2 + x_3 = 2 \\ &x_1 + x_2 + x_4 = 8 \\ &x_1 - x_2 + x_5 = 4 \\ &x_i \geq 0 \quad i = 1, \ldots, 5 \end{aligned} \tag{8}$$

From LP in standard form we have:

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}$$

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis and corresponding basic solutions enumeration:

(a) Basic variables: x_1, x_2, x_3.

$$B_1 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad x_B^1 = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix}$$

(b) Basic variables: x_1, x_2, x_5.

$$B_2 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \quad x_B^2 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

(c) Basic variables: x_1, x_3, x_4.

$$B_3 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad x_B^3 = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$$

(d) Basic variables: x_2, x_4, x_5.

$$B_8 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad x_B^8 = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$$

(e) Basic variables: x_3, x_4, x_5.

$$B_9 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad x_B^9 = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}$$

The following are basis but the corresponding basic solutions are not BFS (basic feasible solution).

(f) Basic Variables: x_1, x_3, x_5

$$B_4 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad x_B^4 = \begin{bmatrix} 8 \\ 10 \\ -4 \end{bmatrix}$$

(g) Basic variables: x_1, x_4, x_5

$$B_5 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad x_B^5 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}$$

(h) Basic variables: x_2, x_3, x_4

$$B_6 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \quad x_B^6 = \begin{bmatrix} -4 \\ 6 \\ 12 \end{bmatrix}$$

(i) Basic variables: x_2, x_3, x_5

$$B_7 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad x_B^7 = \begin{bmatrix} 8 \\ -6 \\ 12 \end{bmatrix}$$