Randomized Selection

Same start as for deterministic selection

SELECT**(A,i,n)**

if $(n = 1)$
    then return $A[1]$

$p = \text{MEDIAN}(A)$

$L = \{x \in A : x \leq p\}$
$H = \{x \in A : x > p\}$

if $i \leq |L|$
    then SELECT$(L, i, |L|)$
    else SELECT$(H, i - |L|, |H|)$

Choose pivot p randomly.

Randomized Selection

Same start as for deterministic selection

SELECT**(A,i,n)**

if $(n = 1)$
    then return $A[1]$

$p = A[\textsc{Random}(1, n)]$

$L = \{x \in A : x \leq p\}$
$H = \{x \in A : x > p\}$

if $i \leq |L|$
    then $\textsc{Select}(L, i, |L|)$
    else $\textsc{Select}(H, i - |L|, |H|)$

Analysis

$T(n)$ **is now the *expected running time.***

$T(n) = \sum_{x=1}^{n}$ **Pr(partition is x smallest)**$\cdot$(**Running time when partition is x smallest**) .

Using x and $n-x$ as an upper bound of the sizes of the two sides:

$$\begin{aligned} T(n) &\le \sum_{x=1}^{n} \frac{1}{n}\left((T(x) \textbf{ or } T(n-x)) + O(n)\right) \\ &\le \sum_{x=1}^{n} \frac{1}{n}\left(T(\max\{x, n-x\}) + O(n)\right) \\ &\le \left(\frac{1}{n}\right) \sum_{x=1}^{n} \left(T(\max\{x, n-x\})\right) + O(n) \end{aligned}$$

We now rewrite the max term. Notice that as x goes from 1 to n, the term $\max\{x, n-x\}$ takes on the values $n-1, n-2, n-3, \ldots, n/2, n/2, n/2+1, n/2+2, \ldots, n-1, n$. As an overestimate, we say that it takes all the values between $n/2$ and n twice. Thus we substitute and obtain

$$\begin{aligned} T(n) &\le \left(\frac{2}{n}\sum_{x=0}^{n/2} T(n/2+x)\right) + O(n) \\ &= \frac{2}{n}T(n) + \left(\frac{2}{n}\sum_{x=0}^{n/2-1} T(n/2+x)\right) + O(n) \end{aligned}$$

Analysis

$$
\begin{aligned}
T(n) &\leq \left(\frac{2}{n}\sum_{x=0}^{n/2} T(n/2+x)\right) + O(n) \\
&= \frac{2}{n}T(n) + \left(\frac{2}{n}\sum_{x=0}^{n/2-1} T(n/2+x)\right) + O(n)
\end{aligned}
$$

We pulled out the $T(n)$ terms to emphasize them. We might be worried about having $T(n)$ on the right side of the equation, so we will bring it over the left-hand side and obtain

$$
\left(1-\frac{2}{n}\right)T(n) \leq \left(\frac{2}{n}\sum_{x=0}^{n/2-1} T(n/2+x)\right) + O(n) \ .
$$

We now multiply both sides of the inequality by $n/(n-2)$ to obtain:

$$
T(n) \leq \left(\frac{2}{n-2}\sum_{x=0}^{n/2-1} T(n/2+x)\right) + kn^2/(n-2) \ .
$$

We have replaced the $O(n)$ by kn for some constant k before multiplying by $n/(n-2)$. We do this because we will need to for the proof by induction below.

We now have a recurrence in a nice form. $T(n)$ is on the left, and the right has terms of the form $T(x)$ for $x < n$. We can therefore "guess" that $T(n) = O(n)$ and try to prove it. More precisely, we will prove by induction that $T(n) \leq cn$ for some c. Since the recurrence is in the stated form, we can substitute in on the right hand side and obtain

Analysis

$$
\begin{aligned}
T(n) &\leq \left(\frac{2}{n-2}\sum_{x=0}^{n/2-1} T(n/2+x)\right) + kn^2/(n-2) \\
&\leq \left(\frac{2}{n-2}\sum_{x=0}^{n/2-1} c(n/2+x)\right) + kn^2/(n-2) \\
&= \left(\frac{2c}{n-2}\right)((n/2)(n/2) + (n/2-1)(n/2)/2) + kn^2/(n-2) \\
&= \left(\frac{2c}{n-2}\right)\left(3n^2/8 - n/4\right) + kn^2/(n-2) \\
&= \left(\frac{c}{n-2}\right)\left(3n^2/4 - n/2\right) + kn^2/(n-2) \\
&= \frac{1}{n-2}\left((3c/4+k)n^2 - (c/2)n\right) \\
&= \frac{n}{n-2}((3c/4+k)n - (c/2))
\end{aligned}
$$

Looking at this last term, we see that the leading $n/(n-2)$ is slightly larger than 1, so we can upper bound it by, say $7/6$ for $n \geq 14$ (there are many possible choices of upper bounds.) Our goal, remember, is to show that the term multiplying the n is at most c, and as we will see, this suffices.

So we get

$$T(n) \leq (7/6)((3c/4+k)n - (c/2)) \ .$$

Analysis

We can drop the $c/2$ term because

$$T(n) \leq (7/6)\left((3c/4 + k)n - (c/2)\right) \leq (7/6)\left((3c/4 + k)n\right) \quad .$$

If the right hand side is at most cn we are done. Whether it is will depend on the relative values of c and k. Let's write the constraint we want

$$(7/6)\left((3c/4 + k)n\right) \leq cn$$

We can divide both sides by n and solve for c in terms of k. We get

$$c \geq 28k/3$$

So we just choose $c = 28k/3$ and we are done.