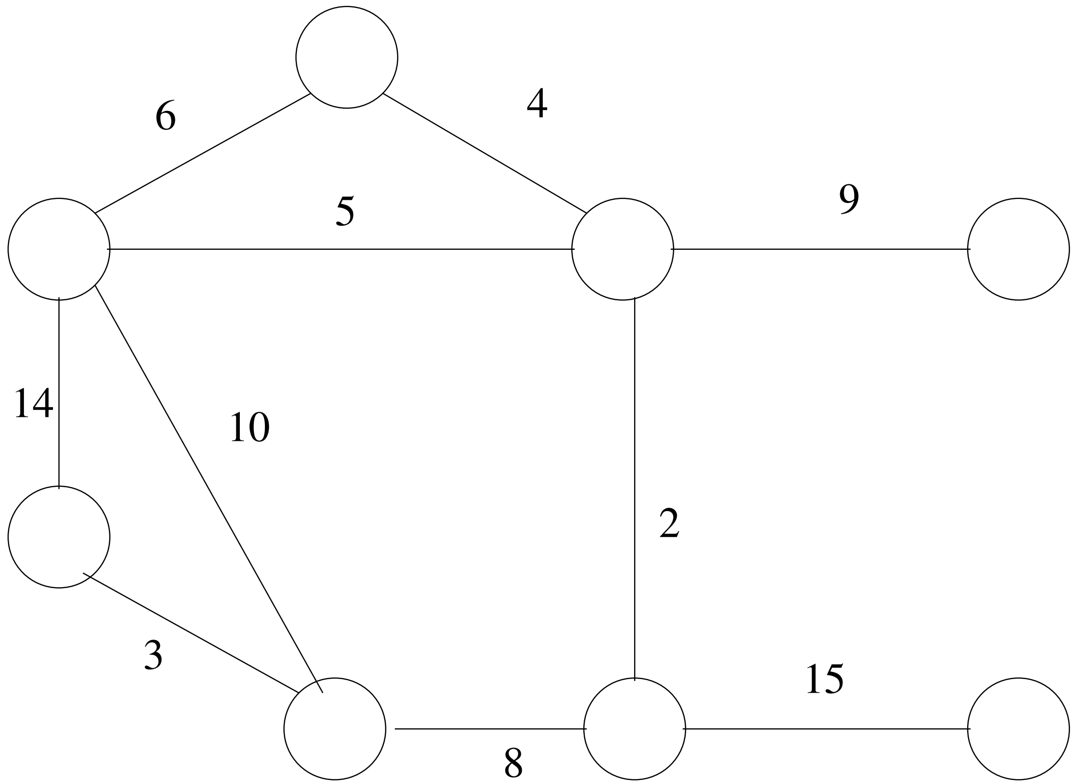
Minimum Spanning Trees

- $G = (V, E)$ **is an undirected graph with non-negative edge weights** $w : E \rightarrow Z^+$

- **We assume wlog that edge weights are distinct**
- **A spanning tree is a tree with** $V - 1$ **edges, i.e. a tree that connects all the vertices.**
- **The total cost (weight) of a spanning tree T is defined as** $w(T) = \Sigma_{e \in T} w(e)$

- **A minimum spanning tree is a tree of minimum total weight.**


6
4
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8

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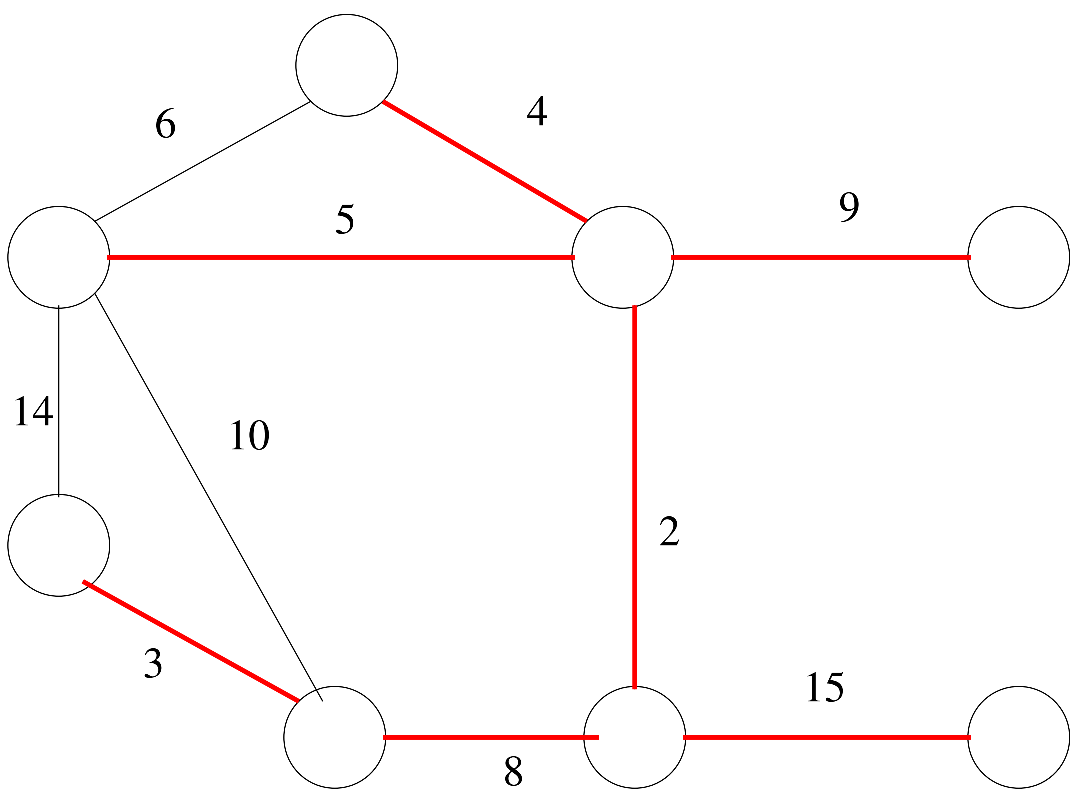
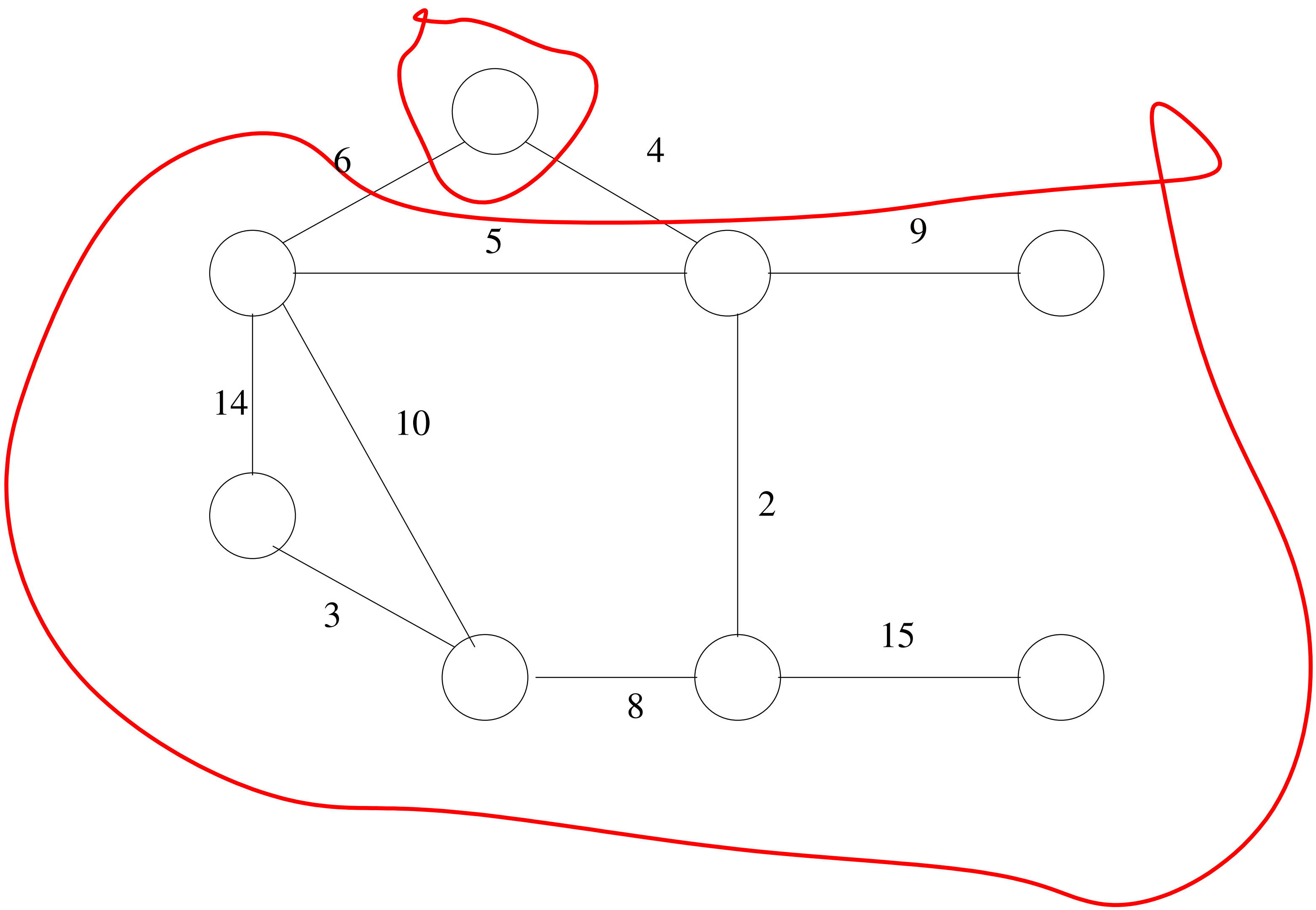


Cuts

- **A cut in a graph is a partition of the vertices into two sets S and T.**
- **An edge (u, v) with $u \in S$ and $v \in T$ is said to cross the cut.**



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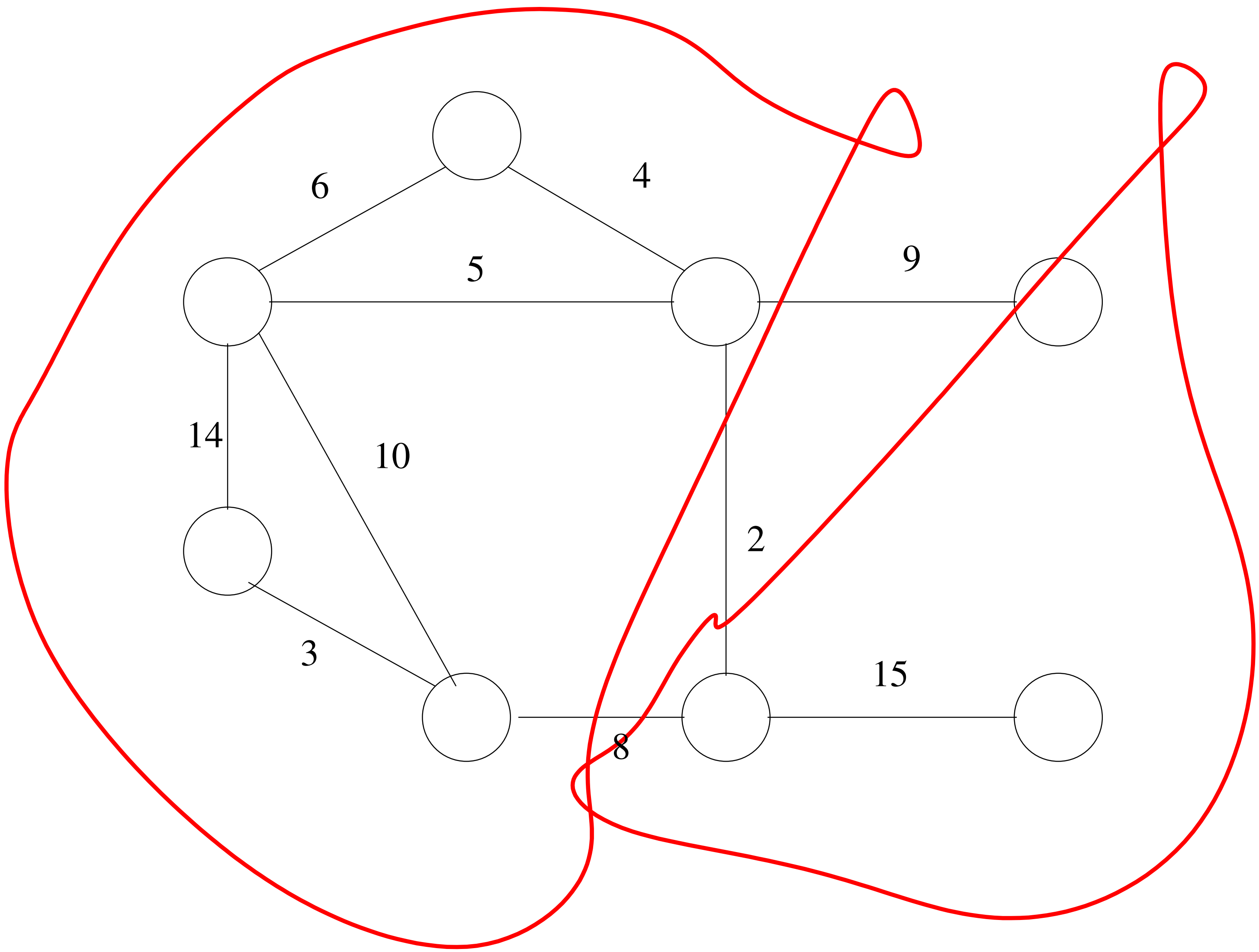

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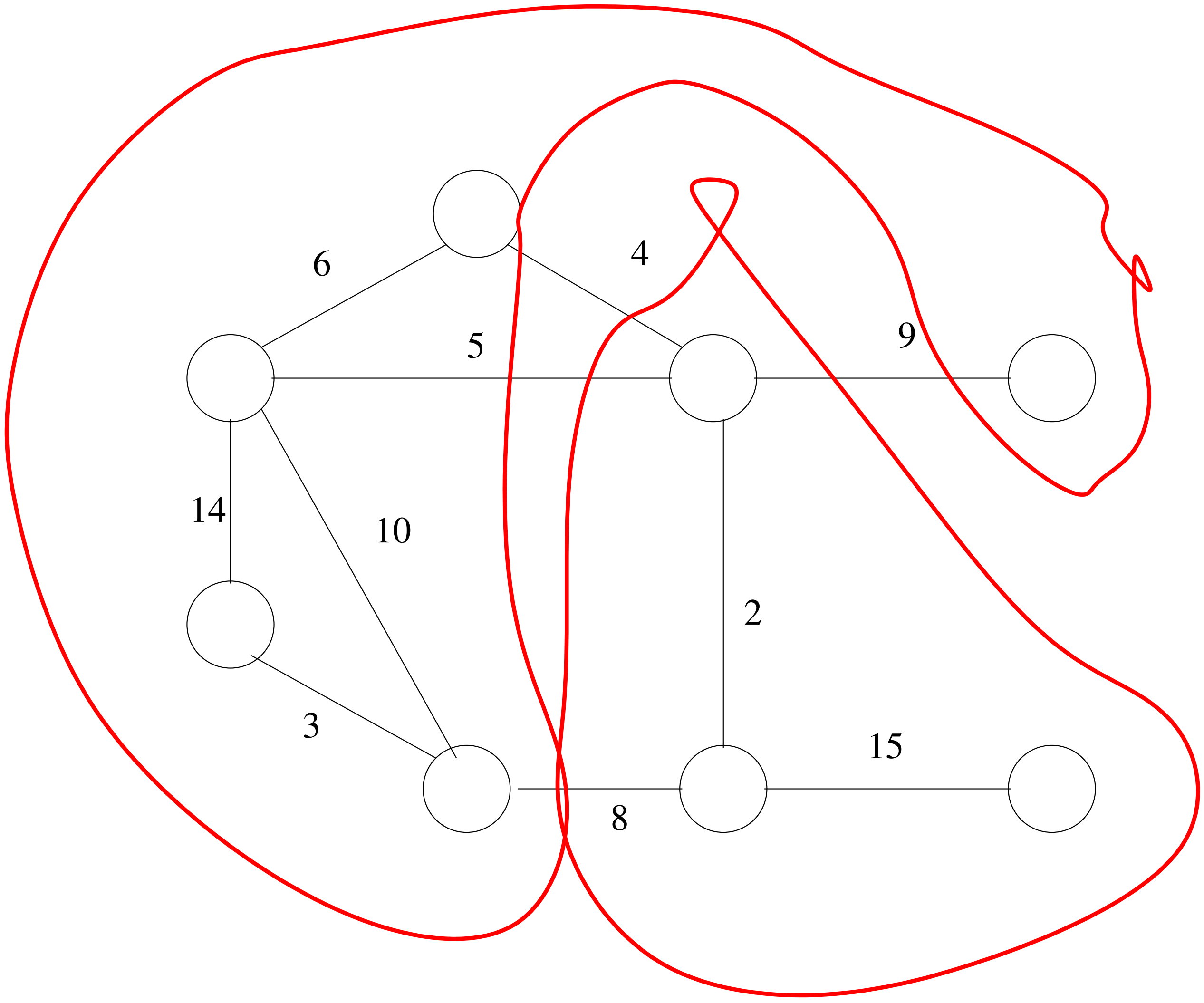


Greedy Property

Recall that we assume all edges weights are unique.

Greedy Property: **The minimum weight edge crossing a cut is in the minimum spanning tree.**

Proof Idea: **Assume not, then remove an edge crossing the cut and replace it with the minimum weight edge.**

Restatement Lemma: **Let $G = (V, E)$ be an undirected graph with edge weights w. Let $A \subseteq E$ be a set of edges that are part of a minimum spanning tree. Let (S, T) be a cut with no edges from A crossing it. Then the minimum weight edge crossing (S, T) can be added to A.**

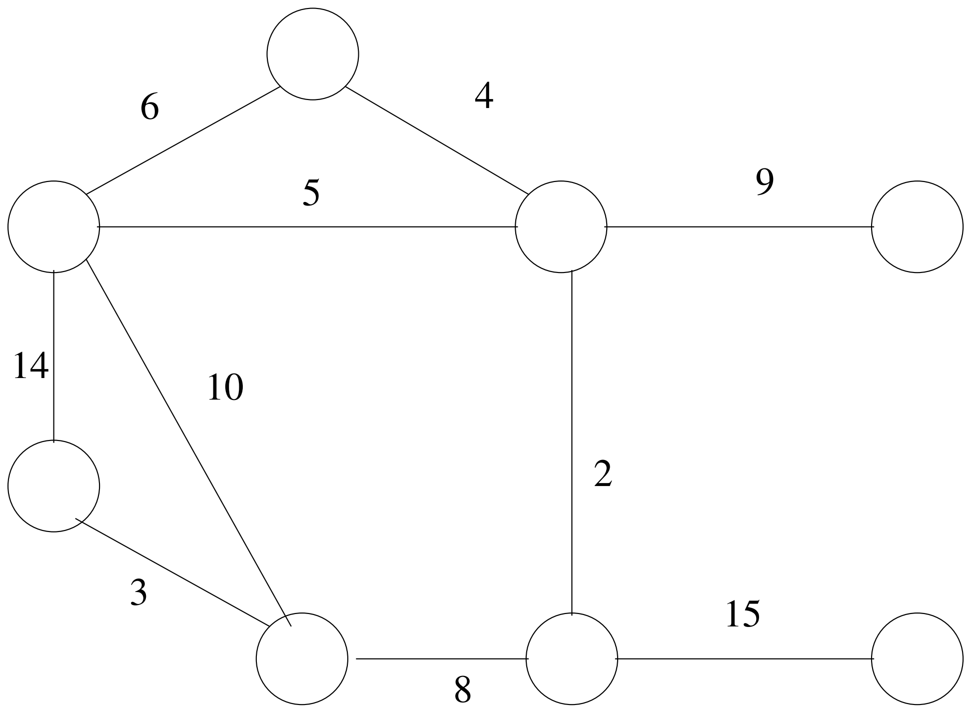
Algorithm Idea: **Repeatedly choose an edge according to the Lemma, add to MST.**

Challenge: **Finding the edge to add.**

Two standard algorithms:

- **Kruskal - consider the edges in increasing order of weight**
- **Prim - start at one vertex and grow the tree.**

Example: Run both algorithms



Kruskal's Algorithm: detailed implementation

Idea: Consider edges in increasing order.

Need: a data structre to maintain the sets of vertices in each component of the current forrest

- MAKE-SET(v) puts v in a set by itself
- FIND-SET(v) returns the name of v's set
- UNION(u, v) combines the sets that u and v are in

MST-Kruskal(G, w)

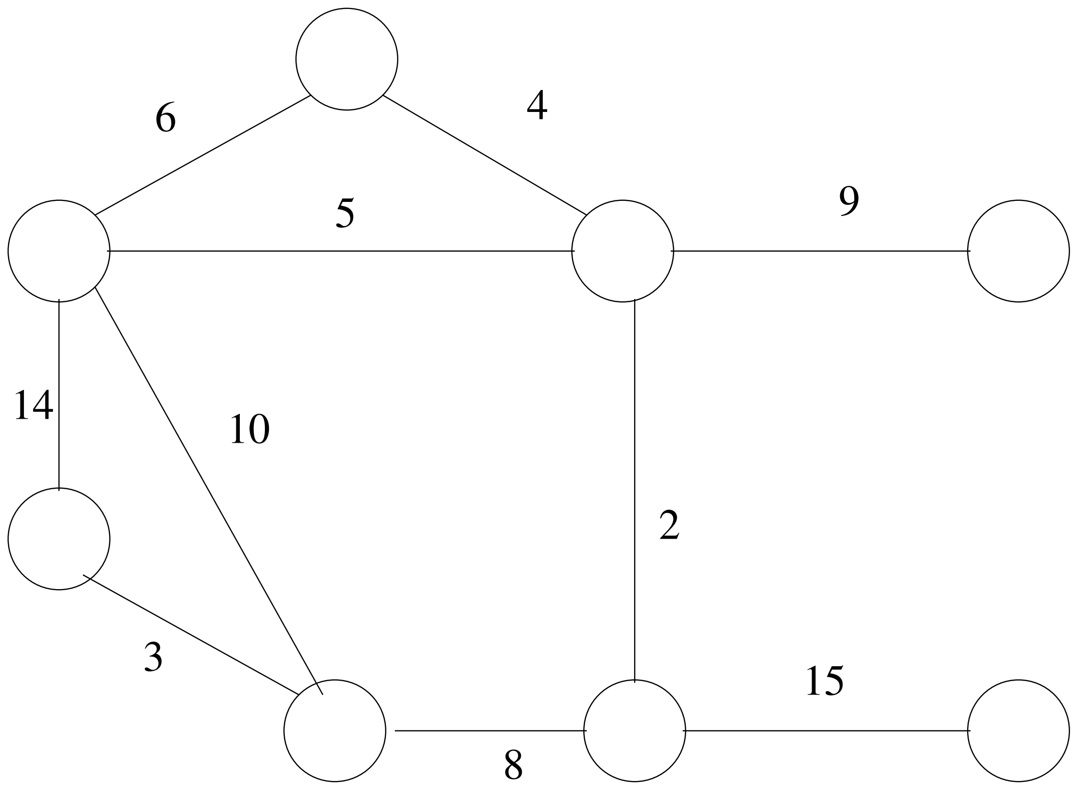
```
A ← ∅
for each vertex v ∈ V[G]
        do MAKE-SET(v)
sort the edges of E into nondecreasing order by weight w
for each edge (u, v) ∈ E, taken in nondecreasing order by weight
        do if FIND-SET(u) ≠ FIND-SET(v)
                then A ← A ∪ {(u, v)}
                        UNION(u, v)
return A
```

Kruskal Running Time

- V Make-Set
- V Union
- E Find-Set

Analysis **After sorting, Kruskal takes** $E \log^* V$ **time (actually slightly better inverse Ackerman time).**

Example



Prim's Algorithm

Idea: Grow the MST from one node going out

Need: a data structure to maintain the edgdes crossing the cut, and choose the minimum. We will maintain, for each vertex, the minimum weight incident edge crossing the cut

- INSERT(v) puts v in the structure
- EXTRACT-MIN(v) finds and returns the node with minimum key value
- DECREASE-KEY(v, δ) updates (decreases) the key of v to δ

MST-Prim(G, w, r)

```
for each u ∈ V[G]
      do key[u] ← ∞
         π[u] ← NIL
         INSERT(u)
key[r] ← 0 ; DECREASE-KEY(r, 0)
while Q ≠ ∅
      do u ← EXTRACT-MIN(Q)
         for each v ∈ Adj[u]
              do if v ∈ Q and w(u, v) < key[v]
                    then π[v] ← u
                         key[v] ← w(u, v)
                         DECREASE-KEY(v, key[v])
```

Analysis

Op	**Heap**	**Fibonacci Heap (amortized)**
V INSERT	$\lg V$	$\lg V$
V EXTRACT-MAIN	$\lg V$	$\lg V$
E DECREASE-KEY	$\lg V$	1
Total	$O(E \lg V)$	$O(E + V \lg V)$

Example